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TOPICAL REVIEW

Random matrices in 2D, Laplacian growth and operator theory

Mark Mineev-Weinstein[1], **Mihai Putinar**[2] **and Razvan Teodorescu**[3]

[1] MS P365, Los Alamos National Laboratory, Los Alamos, NM 87505, USA
[2] Mathematics Department, UCSB, Santa Barbara, CA 93106, USA
[3] Theoretical Division and the Center for Nonlinear Studies, Los Alamos, NM 87505, USA

E-mail: mariner@lanl.gov, mputinar@math.ucsb.edu and razvan@lanl.gov

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Abstract
Since it was first applied to the study of nuclear interactions by Wigner and Dyson, almost 60 years ago, random matrix theory (RMT) has developed into a field of its own within applied mathematics, and is now essential to many parts of theoretical physics, from condensed matter to high energy. The fundamental results obtained so far rely mostly on the theory of random matrices in one dimension (the dimensionality of the spectrum or equilibrium probability density). In the last few years, this theory has been extended to the case where the spectrum is two dimensional, or even fractal, with dimensions between 1 and 2. In this paper, we review these recent developments and indicate some physical problems where the theory can be applied.

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(Some figures in this article are in colour only in the electronic version)

Contents

1. Introduction

During the second half of last century and continuing through the present, random matrix theory has grown from a special method of theoretical physics, meant to approximate energy levels of complex nuclei [1–7], into a vast mathematical theory with many different applications in physics, computer and electrical engineering. Simply describing all the developments and methods currently employed in this context would result in a monography much more extensive than this review. Therefore, we will only briefly mention topics which are themselves very interesting, but lie beyond the scope of this work.

The applications of random matrix theory (RMT) in physics have been extended from the original subject, spectra of heavy nuclei, to descriptions of large N $SU(N)$ gauge theory [8, 9], critical statistical models in two dimensions [10–12], disordered electronic systems [13–18], quantum chromo-dynamics (QCD) [19, 20], to name only a few. Non-physics applications range from communication theory [21] to stochastic processes out of equilibrium [22, 23] and even more exotic topics [24].

A number of important results, both at theoretical and applied levels, were obtained from the connection between random matrices and orthogonal polynomials, especially in their weighted limit [9, 25–28]. These works explored the relationship between the branch cuts of spectral (Riemann) curves of systems of differential equations and the support of limit measures for weighted orthogonal polynomials. Yet another interesting connection stemming from this approach is with the general (matrix) version of the Riemann–Hilbert problem with finite support [29, 30].

In [31, 32], it was shown that such relationships also hold for the class of normal random matrices. Unlike in previous works, for this ensemble, the support of the equilibrium distribution for the eigenvalues of matrices in the infinite-size limit is two-dimensional, which allows us to interpret it as a growing cluster in the plane. Thus, a direct relation to the class of models known as Laplacian growth (both in the deterministic and stochastic formulations) was derived, with important consequences. In particular, this approach allowed us to study formation of singularities in models of two-dimensional growth. Moreover, these results allowed us to define a proper way of continuing the solution for singular Laplacian growth, beyond the critical point.

From the point of view of the dimensionality of the support for random matrix eigenvalues, it is possible to distinguish between one-dimensional situations (which characterize 1- and 2-matrix models) and two-dimensional situations, like in the case of normal random matrix theory. In fact, very recent results point to intermediate cases, where the support is a set of dimensional between 1 and 2. This situation is very similar to the description of disordered, interacting electrons in the plane, in the vicinity of the critical point which separates localized from de-localized behavior [15]. It is from the perspective of the dimensionality of support for equilibrium measure that we have organized this review.

The paper is structured in the following way: after a brief summary of the main concepts in section 2, we explain the structure of normal random matrices in the limit of infinite size, in section 3. This allows us to connect with planar growth models, of which Laplacian (or harmonic) growth is a main representative. The following two sections give a solid description of the physical (section 4) and mathematical (section 5) structure of harmonic growth. The discretized (or quantized) version of this problem is precisely given by normal random matrices, as we indicate in these sections. Next we present a general scheme for encoding shade functions in the plane into linear data, specifically into a linear bounded Hilbert space operator T with rank-1 self-commutator rank $[T^*, T] = 1$. This line of research goes back to the perturbation and scattering theory of symmetric operators (M G Krein's phase shift function) and to studies related to singular integral operators with a Cauchy kernel type singularity. Multivariate refinements of the 'quantization scheme' we outline in section 5 lie at the foundations of both cyclic (co)homology of operator algebras and of free probability theory. In view of the scope and length of the present survey, we confine ourselves to only outline the surprising link between quadrature domains and such Hilbert space objects.

We conclude with an application of the operator formalism to the description of boundary singular points that are characteristic to Laplacian growth evolution, and a brief overview of other related topics.

2. Random matrix theory in 1D

2.1. The symmetry group ensembles and their physical realizations

Following [33], we reproduce the standard introduction of the symmetry-groups ensemble of random matrices. The traditional ensembles (orthogonal, unitary and symplectic) were introduced mainly because of their significance with respect to symmetries of Hamiltonian operators in physical theories: time-reversal and rotational invariance corresponds to the orthogonal ensemble (which, for Gaussian measures, is naturally abbreviated GOE), while time-reversal alone and rotational invariance alone correspond to the symplectic and unitary ensembles, respectively (GSE and GUE for Gaussian measures).

An invariant measure is defined for each of these ensembles, in the form

$$\mathrm{d}\widetilde{\mu}(M) \equiv P(M)\,\mathrm{d}\mu(M) \equiv Z^{-1}\,\mathrm{e}^{-\mathrm{Tr}[W(M)]}\,\mathrm{d}\mu(M), \tag{1}$$

where M is a matrix from the ensemble, Z is a normalization factor (partition function), $\mathrm{Tr}[W(M)]$ is invariant under the symmetries on the ensemble and $\mathrm{d}\mu(M)$ is the appropriate flat measure for that ensemble: $\prod_{i\leqslant j} \mathrm{d}M_{ij}$ for orthogonal, $\prod_{i\leqslant j} d\,\mathrm{Re}\,M_{ij} \prod_{i<j} d\,\mathrm{Im}\,M_{ij}$ for unitary and $\prod_{i\leqslant j} \mathrm{d}M_{ij}^{(0)} \prod_{k=1}^{3} \prod_{i<j} \mathrm{d}M_{ij}^{(k)}$ for symplectic (where each matrix element is an element of the real Klein group, $M_{ij} = M_{ij}^{(0)} \cdot 1 + \sum_{k=1}^{3} M_{ij}^{(k)} \cdot \sigma_k$). Correspondingly, to

each of these ensembles, a parameter β, indicating the number of independent real parameters necessary to describe the pair of values M_{ij}, M_{ji}, is introduced, with values $\beta = 1, 2, 4$ for orthogonal, unitary and symplectic ensembles, respectively.

The invariance under transformations from the appropriate symmetry group leads to the following simplification of the measure: for any of these ensembles, the generic matrix M can be diagonalized by a transformation $M = U^{-1}\Lambda U$, with U from the same group, and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$. The Jacobian of the transformation $M \to \Lambda, U$ (where U is said to carry the 'angular' degrees of freedom of M) is $J = \prod_{i<j} |\lambda_i - \lambda_j|^\beta = |\Delta(\Lambda)|^\beta$, with Δ the Vandermonde determinant. The angular degrees of freedom can be integrated out (a trivial redefinition of the normalization factor), giving the simplified measure

$$\rho(\lambda_1, \ldots, \lambda_N) \prod_{i=1}^{N} \mathrm{d}\lambda_i = Z^{-1}\,\mathrm{e}^{\mathrm{Tr}[W(\Lambda)]} |\Delta(\Lambda)|^\beta \prod_{i=1}^{N} \mathrm{d}\lambda_i. \tag{2}$$

For example, in the case of Gaussian measure $W(M) = -M^2$, the joint probability distribution function of eigenvalues, ρ, becomes (up to normalization)

$$\rho(\lambda_1, \ldots, \lambda_N) = \exp\left[-\sum_{i=1}^{N} \lambda_i^2 + \beta \sum_{i<j} \log|\lambda_i - \lambda_j|\right]. \tag{3}$$

Clearly, this procedure is useful only if we are interested in computing expectation values of quantities which depend only on the distribution of eigenvalues, and not of the angular degrees of freedom. This is indeed the case for all situations of interest.

The next standard transformation (which we discuss for the case of unitary ensemble, $\beta = 2$) that is performed on the measure uses the well-known property of Vandermonde determinant $\Delta(\Lambda) = \det\left[\lambda_i^{j-1}\right]_{1\leqslant i,j\leqslant N}$. Because of standard determinantal identities, this is equivalent with replacing each monomial λ_i^{j-1} by a *monic* polynomial of the same order, $P_{j-1}(\lambda_i) = \lambda_i^{j-1} + \cdots$. Finally, these polynomials may be chosen to be orthogonal with respect to the measure $\mathrm{e}^{W(\lambda)}$, giving for the p.d.f. of eigenvalues the expression

$$\rho(\lambda_1, \ldots, \lambda_N) = |\det[P_{j-1}(\lambda_i)\,\mathrm{e}^{W(\lambda_i)/2}]|^2, \tag{4}$$

which is simply the absolute value squared of the wavefunction of the ground state for N electrons in the external potential W. As we shall see, this kind o physical interpretation may be generalized to the case of matrix ensembles with two-dimensional support of eigenvalues.

Generalizations of group ensembles. Recently, various generalizations were proposed in order to extend the theory for ensembles of matrices which are not associated with symmetry groups. In particular, ensembles of matrices which may be reduced to a tridiagonal form (instead of standard diagonal) by a transformation which eliminates 'angular' degrees of freedom were introduced in [34]. As an interesting consequence, many results carry over to this case, while the parameter β is allowed to take any positive real value.

2.2. *Critical ensembles*

In this section we explain how, using properly chosen non-Gaussian measures, it is possible to construct ensembles of Hermitian matrices (corresponding again to the unitary symmetry) which are in a sense, critical, i.e. for which a continuum limit ($N \to \infty$) may be defined. The discussion relies on the formulation based on orthogonal polynomials indicated above, and it follows (at a more elementary level) the general theory of Saff and Totik [26].

2.2.1. General formalism. Let $\mathrm{d}\mu(x) = \mathrm{e}^{W(x)}\,\mathrm{d}x$ be a well-defined measure on the real axis, $W(x) \to -\infty$ as $|x| \to \infty$, and $P_n^{(1)}(x)$ the corresponding family of orthogonal polynomials

$$\int_{-\infty}^{\infty} P_n^{(1)}(x) P_m^{(1)}(x)\,\mathrm{d}\mu(x) = \delta_{nm}. \tag{5}$$

Orthonormal functions are obtained through $\psi_n(x) = P_n(x)\,\mathrm{e}^{W(x)/2}$, which are orthogonal with respect to the flat measure on $\mathbb{R}$. We consider a deformation of this ensemble through a positive real parameter $\lambda \geqslant 1$, so that $\mathrm{d}\mu_\lambda(x) = \mathrm{e}^{\lambda W(x)}\,\mathrm{d}x$ and

$$\int_{-\infty}^{\infty} P_n^{(\lambda)}(x) P_m^{(\lambda)}(x)\,\mathrm{d}\mu_\lambda(x) = \delta_{nm}. \tag{6}$$

Clearly, if $W(x)$ is a monomial of degree k, the deformation amounts to a simple rescaling

$$P_n^{\lambda}(x) = \lambda^{1/2k} P_n^{(1)}(\lambda^{1/k} x). \tag{7}$$

The first non-trivial example is a quartic polynomial of the type

$$W(x) = -(x^2 + gx^4), \qquad g > 0, \tag{8}$$

for which the deformation in not a simple rescaling. In this case, it is possible to consider a special limit $n \to \infty$, $\lambda \to \infty$, $\lambda \to nr_c$, where r_c is a constant. As we will see, for a specific value of r_c, this limit yields a special asymptotic behavior of the orthonormal functions $\psi_n(x)$. However, even for the simplest, trivial monomial (a Gaussian), which yields the Hermite polynomials, the asymptotic behavior of the orthogonal functions is non-trivial, in the sense that there are no known good approximations for the case $r_c = O(1)$.

Generically, in this large n, λ limit, we can ask where the wavefunction $\psi_n(x)$ will reach its maximum value, in the saddle point approximation,

$$\max_{|x|} \partial_x |\psi_n(x)| = 0, \tag{9}$$

giving

$$\partial_x \left[\sum_{i=1}^{n} \log(x - \xi_i) - nr_c + W(x) \right] = 0, \tag{10}$$

so that

$$-r_c W'(x) = \frac{2}{n} \sum_{i=1}^{n} \frac{1}{x - \xi_i}, \tag{11}$$

where ξ_i, $i = 1, \ldots, n$ are the roots of the n thpolynomial.

Let

$$\omega(z) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\xi_i - z}, \tag{12}$$

multiply (11) by $(\xi_i - z)^{-1}$ and sum over i, and obtain

$$\omega^2(z) - r_c W'(z)\omega(z) = -\frac{r_c}{n} \sum_{i=1}^{n} \frac{W'(z) - W'(\xi_i)}{z - \xi_i}. \tag{13}$$

Equation (13) can be solved in the large-n limit by assuming that the roots will be distributed with density $\rho(\xi)$ on some compact (possibly disconnected) set $I \subset \mathbb{R}$. Defining

$$R(z) = -\frac{4}{r_c} \int_I \frac{W'(z) - W'(\xi)}{z - \xi} \rho(\xi)\,\mathrm{d}\xi, \tag{14}$$

we obtain

$$\omega^2(z) - r_c W'(z)\omega(z) + \left(\frac{r_c}{2}\right)^2 R(z) = 0. \tag{15}$$

The proper solution of (13) (considering the behavior at ∞ of the function $\omega(z)$) is

$$\omega(z) = \frac{r_c}{2}\left[W'(z) + \sqrt{(W'(z))^2 - R(z)}\right], \tag{16}$$

and (since the function $\omega(z)$ is the Cauchy transform of the density $\rho(x)$) it gives us the asymptotic distribution of zeros as

$$\rho(x) = \frac{1}{2\pi \mathrm{i}}[\omega(x+\mathrm{i}0) - \omega(x-\mathrm{i}0)]. \tag{17}$$

Finally, to obtain the asymptotic form of wavefunctions $\psi_n(x)$, we can write

$$n^{-1}\log\psi_n(x) \to \int \rho(\xi)\log(x-\xi)\,\mathrm{d}\xi - \frac{r_c}{2} + W(x). \tag{18}$$

2.2.2. Continuum limit and integrable equations. There are two related problems for the large-n limit of deformed ensembles described in the previous section. The first is determination of the support of zeros I; the second is the scaling behavior of the orthogonal functions $\psi_n(x)$. In general, the limiting support I may consist of several disconnected segments I_k, $I = \cup_{k=1}^{k=d} I_k$. In the simplest case, it is just one interval $I = [a, b] \subset \mathbb{R}$. In this section we indicate how to determine this support as well as the density $\rho(x)$, and what this yields for the orthogonal functions.

Let the function $W(x)$ be a polynomial of even degree d. From (14) we see that $R(z)$ is a polynomial of degree $d-2$, and therefore solution (16) has generically $2(d-1)$ branch points. Thus, the function $\omega(z)$ typically has $d-1$ branch cuts, which constitute the disconnected support of distribution $\rho(z)$.

We are interested in a special case, when $d-2$ of these cuts degenerate into double points, and there is a single interval $[a, b]$ which is the support of $\rho(z)$. This special case is called *critical* and it provides new asymptotic limits for the orthogonal functions. We will also refer to this solution as the 'single-cut' solution.

From the equation

$$\omega(x+\mathrm{i}0) + \omega(x-\mathrm{i}0) = r_c W'(x), \tag{19}$$

we obtain for the single-cut solution

$$\omega(z) = -\frac{r_c\sqrt{(z-a)(z-b)}}{2\pi}\int_a^b \frac{W'(\xi)}{\sqrt{(b-\xi)(\xi-a)}}\frac{\mathrm{d}\xi}{\xi - z}.$$

The large $|z|$ behavior of this function is known from the continuum limit of (12), and it implies the absence of regular terms in the Laurent expansion

$$\omega(z) = -\frac{1}{z} + O(z^{-2}), \tag{20}$$

so that we impose the conditions

$$0 = \int_a^b \frac{W'(\xi)}{\sqrt{(b-\xi)(\xi-a)}}\,\mathrm{d}\xi, \tag{21}$$

$$2\pi = -r_c\int_a^b \frac{\xi W'(\xi)}{\sqrt{(b-\xi)(\xi-a)}}\,\mathrm{d}\xi. \tag{22}$$

Gaussian measure and the Hermite polynomials. Let $d = 2$ and $-W(x) = ax^2, a > 0$. Then conditions (21), (22) give a symmetric support $[-b, b]$ where $b^2 = 2/(ar_c)$.

More generally, using the saddle point equation for $|\psi_n(x)|$ at $x = a, b$ and (18), we conclude that

$$\frac{\log \psi_n(b+\zeta)}{n} = C_b - \frac{r_c}{2}\int_0^{\zeta} \sqrt{[W'(b+\eta)]^2 - R}\,\mathrm{d}\eta. \tag{23}$$

Since the integrand behaves like $\eta^{d-3/2}$, we obtain

$$\psi_n(b+\zeta) = \psi_n(b)\exp\left[-\frac{nr_c}{2d-1}\zeta^{d-1/2}\right]. \tag{24}$$

We immediately conclude that for $d = 2$ (Hermite polynomials), the asymptotic behavior is given by the Airy function, $\exp z^{3/2}$. The full scaling is achieved by considering the region around the end-point b, of order $\zeta = O(n^{-2/(2d-1)})$. Then we obtain

$$\psi_n(b+\tilde{\zeta}n^{-\frac{2}{2d-1}}) \sim \exp\left[-\frac{r_c}{2d-1}\tilde{\zeta}^{d-1/2}\right]. \tag{25}$$

2.2.3. Scaled limits of orthogonal polynomials and equilibrium measures. The distribution of eigenvalues investigated in the previous sections illustrates the general approach developed by Saff and Totik [26] for holomorphic polynomials orthogonal on curves in the complex plane. We sketch here the more general result because of its relevance to the main topic of this review.

Given a set $\Sigma \in \mathbb{C}$ and a properly-defined measure on it $w(z) = \mathrm{e}^{-Q(z)}$, we construct the holomorphic orthogonal polynomials $P_n(z)$, with respect to w. We then pose the question of finding the 'extremal' measure (its support S_w and density μ_w), such that the F-functional $F(K) \equiv \log \mathrm{cap}(K) - \int Q\,\mathrm{d}\omega_K$, with $\mathrm{cap}(K)$ and ω_K the capacity, respectively the equilibrium measure of the set K, is maximized by S_w. Furthermore, μ_w satisfies energy and capacity constraints on S_w.

The remarkable fact noted in [26] is that if the extremal value $F(S_w)$ is approximated by the weighted *monic* polynomials $\tilde{P}_n(z)$ as $(\|w^n\tilde{P}_n\|_{\Sigma}^{*})^{1/n} \to \exp(-F_w)$ (where we use the weak star norm), then the asymptotic zero distribution of $\tilde{P}_n$ gives the support S_w. Hence, (17) may be interpreted as giving both the support of the extremal measure (labeled ρ in this formula), as well as its actual density.

The extremal measure has the physical interpretation of the 'smallest' equilibrium measure which gives a prescribed logarithmic potential at infinity. According to the concept of 'sweeping' (or 'balayage', see [26]), the extremal measure is obtained as a limit of the process, under the constraints imposed on the total mass and energy of the measure. As we have shown in this section, for the case of 1D measures, this extreme case is given by weighted limits of orthogonal polynomials.

3. Random matrix theory in higher dimensions

In this section, we show how to generalize the concepts of equilibrium measure, extremal measure, and their relations to orthogonal polynomials and ensembles of random matrices, in the case of two-dimensional support. The applications of this theory to planar growth processes will be discussed in the following two sections.

3.1. The Ginibre–Girko ensemble

We begin with a brief discussion on the oldest and simplest ensemble of random matrices with planar support. The ensemble of complex, $N \times N$ random matrices with identical, independent, zero-mean Gaussian-distributed entries, was first studied by Ginibre in 1965 [35], and then it was generalized for nonzero mean Gaussian by Girko in 1985 [36]. Consider $N \times N$ random matrices with eigenvalues $z_k \in \mathbb{C}$, and joint p.d.f.

$$\mathrm{d}P_N \sim \prod_{1 \leqslant i<j \leqslant N} |z_i - z_j|^2 \prod_{1 \leqslant k \leqslant N} \mu_N(z_k), \tag{26}$$

where $\mu_N(z_k) = \mathrm{e}^{-N|z_k|^2}\,\mathrm{d}\,\mathrm{Re}\, z_k\, \mathrm{d}\,\mathrm{Im}\, z_k$. Then, in the large-$N$ limit, the measure $\frac{1}{N}\sum_k \delta(z - z_k)$ converges weakly to the uniform measure on the unit disk. This is known as the circular law. If the exponent of the pure Gaussian is perturbed by a quadratic term, the result holds for a corresponding elliptical domain, giving the Elliptical Law. The same limiting curves (circular and elliptical) describe the graph of the distribution of *real* eigenvalues for Hermitian ensembles, with pure and perturbed Gaussian measures. In that case, the laws are known as Wigner–Dyson [1, 4] and Marchenko–Pastur, respectively (although the last one was originally derived for covariance matrices built from sparse regression matrices [37]).

Extensions and exceptions from the circular and elliptical laws were found by relaxing the conditions of the theorems. In particular, deviations from uniformity for angular statistics in the case of Gaussian measure were derived in [38], while the case of heavy-tail distributions was investigated in [39, 40] and subsequent publications.

3.2. Normal matrix ensembles

A special case of matrices with complex eigenvalues is given by *normal* matrices. A matrix M is called normal if it commutes with its Hermitian conjugate: $[M, M^\dagger] = 0$, so that both M and $M^\dagger$ can be diagonalized simultaneously. The statistical weight of the normal matrix ensemble is given through a general potential $W(M, M^\dagger)$ [41],

$$\mathrm{e}^{\frac{1}{\hbar}\,\mathrm{tr}\, W(M,M^\dagger)}\,\mathrm{d}\mu(M). \tag{27}$$

Here $\hbar$ is a parameter, and the measure of integration over normal matrices is induced by the flat metric on the space of all complex matrices $d_C M$, where $d_C M = \prod_{ij} \mathrm{d}\,\mathrm{Re}\, M_{ij}\mathrm{d}\,\mathrm{Im}\, M_{ij}$. Using a standard procedure, one passes to the joint probability distribution of eigenvalues of normal matrices $z_1, \ldots, z_N$, where N is the size of the matrix,

$$\frac{1}{N!\tau_N}|\Delta_N(z)|^2 \prod_{j=1}^{N} \mathrm{e}^{\frac{1}{\hbar}W(z_j,\bar{z}_j)} d^2 z_j. \tag{28}$$

Here $d^2 z_j \equiv \mathrm{d}x_j\,\mathrm{d}y_j$ for $z_j = x_j + \mathrm{i}y_j$, $\Delta_N(z) = \det\left(z_j^{i-1}\right)_{1\leqslant i,j\leqslant N} = \prod_{i>j}^N (z_i - z_j)$ is the Vandermonde determinant, and

$$\tau_N = \frac{1}{N!}\int |\Delta_N(z)|^2 \prod_{j=1}^{N} \mathrm{e}^{\frac{1}{\hbar}W(z_j,\bar{z}_j)}\,\mathrm{d}^2 z_j \tag{29}$$

is a normalization factor, the partition function of the matrix model (a τ-function).

A particularly important special case arises if the potential W has the form

$$W = -|z|^2 + V(z) + \overline{V(z)}, \tag{30}$$

where $V(z)$ is a holomorphic function in a domain which includes the support of eigenvalues (see also a comment in the end of section 3.4 about a proper definition of the ensemble

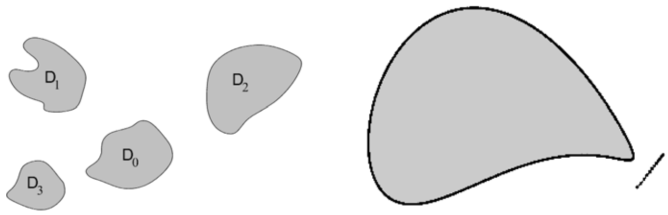

Figure 1. A support of eigenvalues consisting of four disconnected components (left). The distribution of eigenvalues for potential $V(z) = -\alpha \log(1 - z/\beta) - \gamma z$ (right).

with this potential). In this case, a normal matrix ensemble gives the same distribution as a general complex matrix ensemble. A general complex matrix can be decomposed as $M = U(Z + R)U^{\dagger}$, where U and Z are unitary and diagonal matrices, respectively, and R is an upper triangular matrix. The distribution (28) holds for the elements of the diagonal matrix Z which are eigenvalues of M. Here we mostly focus on the special potential (30), and also assume that the field

$$A(z) = \partial_z V(z) \tag{31}$$

is a globally defined meromorphic function.

3.3. Droplets of eigenvalues

In the large-N limit ($\hbar \to 0$, $N\hbar$ fixed), the eigenvalues of matrices from the ensemble densely occupy a connected domain D in the complex plane, or, in general, several disconnected domains. This set (called the support of eigenvalues) has sharp edges (figure 1). We refer to the connected components D_α of the domain D as *droplets*.

For algebraic domains (the definition follows) the eigenvalues are distributed with the density $\rho = -\frac{1}{4\pi}\Delta W$, where $\Delta = 4\partial_z\partial_{\bar{z}}$ is the 2D Laplace operator [31]. For the potential (30) the density is uniform. The shape of the support of eigenvalues is the main subject of this section. For example, if the potential is Gaussian [35],

$$A(z) = 2t_2 z, \tag{32}$$

the domain is an ellipse. If A has one simple pole,

$$A(z) = -\frac{\alpha}{z - \beta} - \gamma \tag{33}$$

the droplet (under certain conditions discussed below) has the profile of an aircraft wing given by the Joukowsky map (figure 1). If A has one double pole (say, at infinity)

$$A(z) = 3t_3 z^2, \tag{34}$$

the droplet is a hypotrochoid. If A has two or more simple poles, there may be more than one droplet. This support and density represent the equilibrium solution to an electrostatic problem, as we will indicate in a later section.

3.4. Orthogonal polynomials and distribution of eigenvalues

Define the exact N-particle wavefunction (up to a phase) by

$$\Psi_N(z_1, \ldots, z_N) = \frac{1}{\sqrt{N!\tau_N}} \Delta_N(z)\, \mathrm{e}^{\sum_{j=1}^{N} \frac{1}{2\hbar} W(z_j, \bar{z}_j)}. \tag{35}$$

The joint probability distribution (28) is then equal to $|\Psi(z_1, \ldots, z_N)|^2$.

Let the number of eigenvalues (particles) increase while the potential stays fixed. If the support of eigenvalues is simply connected, its area grows as $\hbar N$. One can describe the evolution of the domain through the density of particles

$$\rho_N(z) = N \int |\Psi_N(z, z_1, z_2, \ldots, z_{N-1})|^2 \, \mathrm{d}^2 z_1 \cdots \mathrm{d}^2 z_{N-1}, \tag{36}$$

where Ψ_N is given by (35).

We introduce a set of orthonormal one-particle functions on the complex plane as matrix elements of transitions between N and $(N + 1)$-particle states,

$$\frac{\psi_N(z)}{\sqrt{N+1}} = \int \Psi_{N+1}(z, z_1, z_2, \ldots, z_N) \overline{\Psi_N(z_1, z_2, \ldots, z_N)} \, \mathrm{d}^2 z_1 \cdots \mathrm{d}^2 z_N. \tag{37}$$

Then the rate of the density change is

$$\rho_{N+1}(z) - \rho_N(z) = |\psi_N(z)|^2. \tag{38}$$

The proof of this formula is based on the representation of the ψ_n through holomorphic biorthogonal polynomials $P_n(z)$. Up to a phase

$$\psi_n(z) = \mathrm{e}^{\frac{1}{2\hbar} W(z,\bar{z})} P_n(z), \qquad P_n(z) = \sqrt{\frac{\tau_n}{\tau_{n+1}}} z^n + \cdots. \tag{39}$$

The polynomials $P_n(z)$ are biorthogonal on the complex plane with the weight $\mathrm{e}^{W/\hbar}$,

$$\int \mathrm{e}^{W/\hbar} P_n(z) \overline{P_m(z)} \, \mathrm{d}^2 z = \delta_{mn}. \tag{40}$$

The proof of these formulae is standard in the theory of orthogonal polynomials. Extension to the biorthogonal case adds no difficulties.

We note that, with the choice of potential (30), the integral representation (40) has only a formal meaning, since the integral diverges unless the potential is Gaussian. A proper definition of the wavefunctions goes through recursive relations (53), (54) which follow from the integral representation. The same comment applies to the τ-function (29). The wavefunction is not normalized everywhere in the complex plane. It may diverge at the poles of the vector potential field.

3.5. Wavefunctions, recursions and integrable hierarchies

In order to illustrate the mathematical connection between this theory and equivalent formulations which we present in section 5, it is necessary to make a digression through the formalism of infinite, integrable hierarchies. In particular, we choose the case of the Kadomtsev–Petviashvilii (KP) hierarchy, and follow the notations in [42].

3.5.1. Pseudo-differential operators. We denote by $\mathcal{A}$ the algebra constructed from differential polynomials of the type $P = \partial^n + u_{n-2}\partial^{n-2} + \cdots + u_1\partial + u_0$, where $\partial = \partial/\partial z$ is a differential symbol with respect to some (complex) variable z, and $u_0, u_1, \ldots, u_{n-2}$ (note: u_{n-1} can be always set to zero) are generically smooth functions in z and (if necessary) other variables $t_1, t_2, \ldots$. On this algebra, we define the ring of pseudo-differential operators $\mathcal{R}$, consisting of (formal) operators defined by the infinite series

$$L = \sum_{-\infty}^{n} c_k \partial^k, \qquad \partial^{-1} \equiv \int \mathrm{d}z, \tag{41}$$

where coefficients are again smooth functions, and the negative powers in the expansions contain *integral* operators. For any such operator, we denote by L_+ the purely differential part and by L_- the remainder of the series

$$L_+ \equiv \sum_0^n c_k \partial^k, \qquad L = L_+ + L_-. \tag{42}$$

Let

$$\mathcal{L} = \partial + u_0 \partial^{-1} + u_1 \partial^{-2} + \cdots \tag{43}$$

be a pseudo-differential operator such that $\mathcal{L}_+ = \partial$. Then, introducing the infinite set of *times* $\boldsymbol{t} = t_1, t_2, \ldots$, such that all coefficients u_k, c_k above are generically functions of $\boldsymbol{t}$, the KP hierarchy has the form

$$\frac{\partial \mathcal{L}}{\partial t_k} = \left[\mathcal{L}_+^k, \mathcal{L}\right], \qquad k = 1, 2, \ldots. \tag{44}$$

More explicitly, we note that the hierarchy consists of the differential equations satisfied by the *coefficients* of the operator $\mathcal{L}$. As a consequence of the compatibility of all the equations in the hierarchy, we have the *zero-curvature equations*

$$\left[\partial_{t_k} - \mathcal{L}_+^k, \partial_{t_p} - \mathcal{L}_+^p\right] = 0, \qquad \forall\, t_k, t_p. \tag{45}$$

3.5.2. Level reductions. The KP hierarchy contains many other known integrable hierarchies, particularly the KdV hierarchy, as reductions to a certain *level* n in the hierarchy. For example, assume that the operator $\mathcal{L}$ satisfies the constraint

$$\mathcal{L}_-^2 = 0, \tag{46}$$

i.e. it is the square root of a *differential* operator L of order 2,

$$\mathcal{L} = L^{1/2}, \qquad L = \partial^2 + 2u_0. \tag{47}$$

Then it follows that for all even powers $n = 2m$, $\mathcal{L}_+^n = \mathcal{L}^n$, so that $\left[\mathcal{L}_+^n, \mathcal{L}\right] = 0$, so there is *no* dependence on the even times $t_2, t_4, \ldots$. This sub-hierarchy is called *level* -2 reduced KP, and the first non-trivial zero-curvature equation of the hierarchy is the famous Korteweg–de Vries equation

$$\mathcal{L}_+^3 \equiv P = \partial^3 + \frac{3}{2}\left[u_0 \partial + \partial u_0\right], \qquad \frac{\partial L}{\partial t_3} = [P, L] \Rightarrow u_{t_3} = 6 u u_z + u_{zzz}, \tag{48}$$

where we have used $u_0 = u$ for clarity.

This formulation of the KdV equation makes use of the notion of *Lax* pair L, P, which is central to the *inverse scattering method* for solving nonlinear integrable differential equations. The idea is quite physical: assume that the operators L, P act on a wavefunction $\psi(x, t)$ such that

$$L\psi = \lambda \psi, \qquad \frac{\partial \psi}{\partial t} = P\psi, \tag{49}$$

where eigenvalues λ form the spectrum of L. Then applying the Lax pair equation to the eigenvalue equation, we obtain $\partial\lambda/\partial t = 0$, i.e. the evolution under these equations leaves the spectrum invariant. This allows us to construct the initial state from the final state, hence the inverse scattering appellation.

3.5.3. Tau functions and Baker–Akhiezer function. At the level of systems of PDE, the τ-function and the Baker–Akhiezer function are introduced, by analogy with the Lax par formulation indicated above, in the following way:

Baker–Akhiezer function. Consider the function $\psi(z, t_1, t_2, \ldots)$ satisfying

$$\mathcal{L}\psi = z\psi, \qquad \frac{\partial \psi}{\partial t_k} = \mathcal{L}_+^k \psi, \qquad \forall\, k \geqslant 1. \tag{50}$$

This is the Baker–Akhiezer function of the KP hierarchy.

Fundamental property of the Baker–Akhiezer function. Let $\phi = 1 + \sum_0^\infty k_i \partial^{-i-1}$ be the 'dressing' operator defined such that $\mathcal{L} = \phi\partial\phi^{-1}$. Also, introduce the function $g(z, t_1, \ldots) = \exp\left[\sum_1^\infty t_k z^k\right]$. Then the Baker–Akhiezer function satisfies

$$\psi = \hat{k}(z) g(z, t_1, \ldots), \qquad \hat{k}(z) = 1 + \sum_0^\infty k_i z^{-i-1},$$

where $\hat{k}$ is the 'scalar' analog of the dressing operator ϕ.

Tau function. Using the notation introduced above, we have the following property:

There exists a function $\tau(z, t_1, \ldots)$ such that

$$\psi(z, t_1, \ldots) = g \cdot \frac{\tau\left(z, t_1 - \frac{1}{z}, t_2 - \frac{1}{2z^2}, \ldots\right)}{\tau(z, t_1, t_2, \ldots)} = g \cdot \frac{\exp\left[\sum_1^\infty -\frac{1}{kz^k}\frac{\partial}{\partial t_k}\right]\tau(z, t_1, \ldots)}{\tau(z, t_1, \ldots)}.$$

Now let us consider the generalized overlap function

$$\psi_N(z, \bar{w}) = \tau_N^{-1} \int \Psi_{N+1}(z, z_1, z_2, \ldots, z_N)\overline{\Psi_{N+1}(w, z_1, z_2, \ldots, z_N)}\, \mathrm{d}^2 z_1 \cdots \mathrm{d}^2 z_N,$$

and expand for $|z|, |w| \to \infty$. We obtain

$$\psi_N(z, \bar{w}) = \frac{(z\bar{w})^N}{\tau_N} \exp\left[\sum -\frac{1}{z^k \bar{w}^p}\frac{\partial}{\partial a_{kp}}\right]\tau_N, \tag{51}$$

where a_{kp} is the corresponding interior bi-harmonic moment. Therefore, we may regard the τ-function and the scaled wavefunction introduced earlier as canonical objects describing an integrable hierarchy. This fact will be illustrated in more detail in the following section.

3.6. Equations for the wavefunctions and the spectral curve

In this section we specify the potential to be of the form (30). It is convenient to modify the exponential factor of the wavefunction. Namely, we define

$$\psi_n(z) = \mathrm{e}^{-\frac{|z|^2}{2\hbar} + \frac{1}{\hbar}V(z)} P_n(z), \qquad \text{and} \qquad \chi_n(z) = \mathrm{e}^{\frac{1}{\hbar}V(z)} P_n(z), \tag{52}$$

where the holomorphic functions $\chi_n(z)$ are orthonormal in the complex plane with the weight $\mathrm{e}^{-|z|^2/\hbar}$. Like traditional orthogonal polynomials, the biorthogonal polynomials P_n (and the corresponding wavefunctions) obey a set of differential equations with respect to the argument z, and recurrence relations with respect to the degree n. Similar equations for two-matrix models are discussed in numerous papers (see, e.g., [43]).

We introduce the L-operator (the Lax operator) as multiplication by z in the basis χ_n,

$$L_{nm}\chi_m(z) = z\chi_n(z) \tag{53}$$

(summation over repeated indices is implied). Obviously, L is a lower triangular matrix with one adjacent upper diagonal, $L_{nm} = 0$ as $m > n + 1$. Similarly, the differentiation ∂_z is represented by an upper triangular matrix with one adjacent lower diagonal. Integrating by parts the matrix elements of the ∂_z, one finds

$$(L^{\dagger})_{nm}\chi_m = \hbar\partial_z\chi_n, \tag{54}$$

where $L^{\dagger}$ is the Hermitian conjugate operator.

The matrix elements of $L^{\dagger}$ are $(L^{\dagger})_{nm} = \bar{L}_{mn} = A(L_{nm}) + \int \mathrm{e}^{\frac{1}{\hbar}W}\bar{P}_m(\bar{z})\partial_z P_n(z)\,\mathrm{d}^2z$, where the last term is a lower triangular matrix. The latter can be written through negative powers of the Lax operator. Writing $\partial_z \log P_n(z) = \frac{n}{z} + \sum_{k>1} v_k(n)z^{-k}$, one represents $L^{\dagger}$ in the form

$$L^{\dagger} = A(L) + (\hbar n)L^{-1} + \sum_{k>1} v^{(k)}L^{-k}, \tag{55}$$

where $v^{(k)}$ and $(\hbar n)$ are diagonal matrices with elements $v_n^{(k)}$ and $(\hbar n)$. The coefficients $v_n^{(k)}$ are determined by the condition that lower triangular matrix elements of $A(L_{nm})$ are cancelled.

In order to emphasize the structure of the operator L, we write it in the basis of the shift operator[4] $\hat{w}$ such that $\hat{w}f_n = f_{n+1}\hat{w}$ for any sequence f_n. Acting on the wavefunction, we have

$$\hat{w}\chi_n = \chi_{n+1}.$$

In the n-representation, the operators $L, L^{\dagger}$ acquire the form

$$L = r_n\hat{w} + \sum_{k\geqslant 0} u_n^{(k)}\hat{w}^{-k}, \qquad L^{\dagger} = \hat{w}^{-1}r_n + \sum_{k\geqslant 0}\hat{w}^k\bar{u}_n^{(k)}. \tag{56}$$

Clearly, acting on χ_n, we have the commutation relation ('the string equation')

$$[L, L^{\dagger}] = \hbar. \tag{57}$$

This is the compatibility condition of equations (53) and (54).

Equations (56) and (57) completely determine the coefficients $v_n^{(k)}$, r_n and $u_n^{(k)}$. The first one connects the coefficients to the parameters of the potential. The second equation is used to determine how the coefficients $v_n^{(k)}$, r_n and $u_n^{(k)}$ evolve with n. In particular, the diagonal part of it reads

$$n\hbar = r_n^2 - \sum_{k\geqslant 1}\sum_{p=1}^{k}\left|u_{n+p}^{(k)}\right|^2. \tag{58}$$

Moreover, we note that all the coefficients can be expressed through the τ-function (29) and its derivatives with respect to parameters of the potential. This representation is particularly simple for r_n: $r_n^2 = \tau_n\tau_{n+1}^{-2}\tau_{n+2}$.

3.6.1. Finite-dimensional reductions. If the vector potential $A(z)$ is a rational function, the coefficients $u_n^{(k)}$ are not all independent. The number of independent coefficients equals the number of independent parameters of the potential. For example, if the holomorphic part of the potential, $V(z)$, is a polynomial of degree d, the series (56) are truncated at $k = d - 1$.

In this case, the semi-infinite system of linear equations (54) and the recurrence relations (53) can be cast in the form of a set of finite-dimensional equations whose coefficients are rational functions of z, one system for every $n > 0$. The system of differential equations generalizes the Cristoffel–Daurboux second-order differential equation valid for orthogonal polynomials. This fact has been observed in recent papers [44, 45] for biorthogonal

[4] The shift operator $\hat{w}$ has no inverse. Below $\hat{w}^{-1}$ is understood as a shift to the left defined as $\hat{w}^{-1}\hat{w} = 1$. Same is applied to the operator L^{-1}. To avoid a possible confusion, we emphasize that although χ_n is a right-hand eigenvector of L, it is not a right-hand eigenvector of L^{-1}.

polynomials emerging in the Hermitian two-matrix model with a polynomial potential. It is applicable to our case (holomorphic biorthogonal polynomials) as well.

In a more general case, when $A(z)$ is a general rational function with $d-1$ poles (counting multiplicities), the series (56) is not truncated. However, L can be represented as a 'ratio',

$$L = K_1^{-1}K_2 = M_2M_1^{-1}, \tag{59}$$

where the operators $K_{1,2}$, $M_{1,2}$ are polynomials in $\hat{w}$,

$$K_1 = \hat{w}^{d-1} + \sum_{j=0}^{d-2} A_n^{(j)}\hat{w}^j, \qquad K_2 = r_{n+d-1}\hat{w}^d + \sum_{j=0}^{d-1} B_n^{(j)}\hat{w}^j \tag{60}$$

$$M_1 = \hat{w}^{d-1} + \sum_{j=0}^{d-2} C_n^{(j)}\hat{w}^j, \qquad M_2 = r_n\hat{w}^d + \sum_{j=0}^{d-1} D_n^{(j)}\hat{w}^j. \tag{61}$$

These operators obey the relation

$$K_1M_2 = K_2M_1. \tag{62}$$

It can be proven that the pair of operators $M_{1,2}$ is uniquely determined by $K_{1,2}$ and vice versa. We note that the reduction (59) is a difference analog of the 'rational' reductions of the Kadomtsev–Petviashvili integrable hierarchy considered in [46].

The linear problems (53), (54) acquire the form

$$(K_2\chi)_n = z(K_1\chi)_n, \qquad \left(M_2^\dagger\chi\right)_n = \hbar\partial_z\left(M_1^\dagger\chi\right)_n. \tag{63}$$

These equations are of *finite order* (namely, of order d), i.e., they connect values of χ_n on $d+1$ subsequent sites of the lattice.

The semi-infinite set $\{\chi_0, \chi_1, \ldots\}$ is then a 'bundle' of d-dimensional vectors

$$\underline{\chi}(n) = (\chi_n, \chi_{n+1}, \ldots, \chi_{n+d-1})^{\mathrm{t}}$$

(the index t means transposition, so $\underline{\chi}$ is a column vector). The dimension of the vector is the number of poles of $A(z)$ plus one. Each vector obeys a closed d-dimensional linear differential equation

$$\hbar\partial_z\underline{\chi}(n) = \mathcal{L}_n(z)\underline{\chi}(n), \tag{64}$$

where the $d \times d$ matrix $\mathcal{L}_n$ is a 'projection' of the operator $L^\dagger$ onto the nth d-dimensional space. Matrix elements of the $\mathcal{L}_n$ are rational functions of z having the same poles as $A(z)$ and also a pole at the point $\overline{A(\infty)}$. (If $A(z)$ is a polynomial, all these poles accumulate to a multiple pole at infinity).

We briefly describe the procedure of constructing the finite-dimensional matrix differential equation. We use the first linear problem in (63) to represent the shift operator as a $d \times d$ matrix $\mathcal{W}_n(z)$ with z-dependent coefficients,

$$\mathcal{W}_n(z)\underline{\chi}(n) = \underline{\chi}(n+1). \tag{65}$$

This is nothing else than rewriting the scalar linear problem in the matrix form. Then the matrix $\mathcal{W}_n(z)$ is to be substituted into the second equation of (63) to determine $\mathcal{L}_n(z)$ (examples follow). The entries of $\mathcal{W}_n(z)$ and $\mathcal{L}_n(z)$ obey the Schlesinger equation, which follows from compatibility of (64) and (65):

$$\hbar\partial_z\mathcal{W}_n = \mathcal{L}_{n+1}\mathcal{W}_n - \mathcal{W}_n\mathcal{L}_n. \tag{66}$$

This procedure has been realized explicitly for polynomial potentials in recent papers [44, 45]. We will work it out in detail for our three examples: $\underline{\chi}(n) = (\chi_n, \chi_{n+1})^{\mathrm{t}}$ for the ellipse (32) and the aircraft wing (33) and $\underline{\chi}(n) = (\chi_n, \chi_{n+1}, \chi_{n+2})^{\mathrm{t}}$ for the hypotrochoid (34).

3.7. Spectral curve

According to the general theory of linear differential equations, the semiclassical (WKB) asymptotics of solutions to equation (64), as $\hbar \to 0$, is found by solving the eigenvalue problem for the matrix $\mathcal{L}_n(z)$ [47]. More precisely, the basic object of the WKB approach is the spectral curve [47] of the matrix $\mathcal{L}_n$, which is defined, for every integer $n > 0$, by the secular equation $\det(\mathcal{L}_n(z) - \tilde{z}) = 0$ (here $\tilde{z}$ means $\tilde{z} \cdot \mathbf{1}$, where $\mathbf{1}$ is the unit $d \times d$ matrix). It is clear that the left-hand side of the secular equation is a polynomial in $\tilde{z}$ of degree d. We define the spectral curve by an equivalent equation

$$f_n(z, \tilde{z}) = a(z) \det(\mathcal{L}_n(z) - \tilde{z}) = 0, \tag{67}$$

where the factor $a(z)$ is added to make $f_n(z, \tilde{z})$ a polynomial in z as well. The factor $a(z)$ then has zeros at the points where poles of the matrix function $\mathcal{L}(z)$ are located. It does not depend on n. We will soon see that the degree of the polynomial $a(z)$ is equal to d. Assume that all poles of $A(z)$ are simple, then zeros of the $a(z)$ are just the $d - 1$ poles of $A(z)$ and another simple zero at the point $\overline{A(\infty)}$. Therefore, we conclude that the matrix $\mathcal{L}_n(z)$ is rather special. For a general $d \times d$ matrix function with the same d poles, the factor $a(z)$ would be of degree d^2.

Note that the matrix $\mathcal{L}_n(z) - \bar{z}$ enters the differential equation

$$\hbar \partial_z |\underline{\psi}(n)|^2 = \underline{\bar{\psi}}(n)(\mathcal{L}_n(z) - \bar{z})\underline{\psi}(n) \tag{68}$$

for the squared amplitude $|\underline{\psi}(n)|^2 = \underline{\psi}^\dagger(n)\underline{\psi}(n) = \mathrm{e}^{-\frac{|z|^2}{\hbar}}|\underline{\chi}(n)|^2$ of the vectors $\underline{\psi}(n)$ built from the orthonormal wavefunctions (39).

The equation of the curve can be interpreted as a 'resultant' of the non-commutative polynomials $K_2 - zK_1$ and $M_2^\dagger - \tilde{z}M_1^\dagger$ (cf [44]). Indeed, the point $(z, \tilde{z})$ belongs to the curve if and only if the linear system

$$\begin{cases} (K_2 c)_k = z(K_1 c)_k & n - d \leqslant k \leqslant n - 1 \\ \left(M_2^\dagger c\right)_k = \tilde{z}\left(M_1^\dagger c\right)_k & n \leqslant k \leqslant n + d - 1 \end{cases} \tag{69}$$

has non-trivial solutions. The system contains $2d$ equations for $2d$ variables $c_{n-d}, \ldots, c_{n+d-1}$. Vanishing of the $2d \times 2d$ determinant yields the equation of the spectral curve. Below we use this method to find the equation of the curve in the examples. It appears to be much easier than the determination of the matrix $\mathcal{L}_n(z)$.

The spectral curve (67) possesses an important property: it admits an antiholomorphic involution. In the coordinates $z, \tilde{z}$ the involution reads $(z, \tilde{z}) \mapsto (\bar{\tilde{z}}, \bar{z})$. This simply means that the secular equation $\det(\bar{\mathcal{L}}_n(\tilde{z}) - z) = 0$ for the matrix $\bar{\mathcal{L}}_n(\tilde{z}) \equiv \overline{\mathcal{L}_n(\bar{\tilde{z}})}$ defines the same curve. Therefore, the polynomial f_n takes real values for $\tilde{z} = \bar{z}$,

$$f_n(z, \bar{z}) = \overline{f_n(z, \bar{z})}. \tag{70}$$

Points of the real section of the curve ($\tilde{z} = \bar{z}$) are fixed points of the involution.

The curve (67) was discussed in recent papers [44, 45] in the context of Hermitian two-matrix models with polynomial potentials. The dual realizations of the curve pointed out in [44] correspond to the antiholomorphic involution in our case. The involution can be proven along the lines of these works. The proof is rather technical and we omit it, restricting ourselves to the examples below. We simply note that the involution relies on the fact that the squared modulus of the wavefunction is real.

We will give a concrete example for the construction of the spectral curve, after a brief but necessary detour through the continuum limit of this problem.

3.7.1. Schwarz function. The polynomial $f_n(z,\bar{z})$ can be factorized in two ways,

$$f_n(z,\bar{z}) = a(z)\big(\bar{z} - S_n^{(1)}(z)\big)\cdots\big(\bar{z} - S_n^{(d)}(z)\big), \tag{71}$$

where $S_n^{(i)}(z)$ are eigenvalues of the matrix $\mathcal{L}_n(z)$, or

$$f_n(z,\bar{z}) = \overline{a(z)}\big(z - \bar{S}_n^{(1)}(\bar{z})\big)\cdots\big(z - \bar{S}_n^{(d)}(\bar{z})\big), \tag{72}$$

where $\bar{S}_n^{(i)}(\bar{z})$ are eigenvalues of the matrix $\bar{\mathcal{L}}_n(\bar{z})$. One may understand them as different branches of a multivalued function $S(z)$ (respectively, $\bar{S}(z)$) on the plane (here we do not indicate the dependence on n, for simplicity of the notation). It then follows that $S(z)$ and $\bar{S}(z)$ are mutually inverse functions,

$$\bar{S}(S(z)) = z. \tag{73}$$

An algebraic function with this property is called *the Schwarz function*. By the equation $f(z, S(z)) = 0$, it defines a complex curve with an antiholomorphic involution. An upper bound for genus of this curve is $g = (d-1)^2$, where d is the number of branches of the Schwarz function. The real section of this curve is a set of all fixed points of the involution. It consists of a number of contours on the plane (and possibly a number of isolated points, if the curve is not smooth). The structure of this set is known to be complicated. Depending on coefficients of the polynomial, the number of disconnected contours in the real section may vary from 0 to $g+1$. If the contours divide the complex curve into two disconnected 'halves', or sides (related by the involution), then the curve can be realized as the *Schottky double* [48] of one of these sides. Each side is a Riemann surface with a boundary.

Let us come back to equation (64). It has d independent solutions. They are functions on the spectral curve. One of them is a physical solution corresponding to biorthogonal polynomials. The physical solution defines the 'physical sheet' of the curve.

The Schwarz function on the physical sheet is a particular root, say $S_n^{(1)}(z)$, of the polynomial $f_n(z,\tilde{z})$ (see (71)). It follows from (55) that this root is selected by the requirement that it has the same poles and residues as the potential A.

3.7.2. The Schottky double. The Schwarz function describes more than just the boundary of clusters of eigenvalues. Together with other sheets it defines a Riemann surface. If the potential $A(z)$ is meromorphic, the Schwarz function is an algebraic function. It satisfies a polynomial equation $f(z, S(z)) = 0$.

The function $f(z,\tilde{z})$, where z and $\tilde{z}$ are treated as two independent complex arguments, defines a Riemann surface with antiholomorphic involution (70). If the involution divides the surface into two disconnected parts, as explained above, the Riemann surface is the *Schottky double* [48] of one of these parts.

There are two complementary ways to describe this surface. One is through the algebraic covering (71), (72). Among d sheets we distinguish a *physical* sheet. The physical sheet is selected by the condition that the differential $S(z)\,\mathrm{d}z$ has the same poles and residues as the differential of the potential $A(z)\,\mathrm{d}z$. It may happen that the condition $\bar{z} = S^{(i)}(z)$ defines a planar curve (or several curves, or a set of isolated points) for branches other than the physical one. We refer to the interior of these planar curves as *virtual* (or unphysical) droplets situated on sheets other than physical.

Another way emphasizes the antiholomorphic involution. Consider a meromorphic function $h(z)$ defined on a Riemann surface with boundaries. We call this surface the front side. The Schwarz reflection principle extends any meromorphic function on the front side to a meromorphic function on the Riemann surface without boundaries. This is done by adding another copy of the Riemann surface with boundaries (a back side), glued to the front

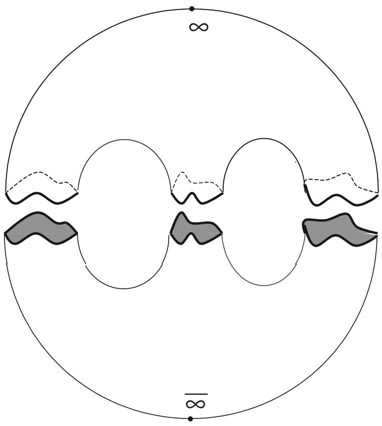

Figure 2. The Schottky double. A Riemann surface with boundaries along the droplets (a front side) is glued to its mirror image (a back side).

side along the boundaries, figure 2. The value of the function h on the mirror point on the back side is $h(\overline{S(z)})$. The copies are glued along the boundaries: $h(z) = h(\overline{S(z)})$ if the point z belongs to the boundary. The same extension rule applies to differentials. Having a meromorphic differential $h(z)\,\mathrm{d}z$ on the front side, one extends it to a meromorphic differential $h(\overline{S(z)})d\overline{S(z)}$ on the back side.

This definition can be applied to the Schwarz function itself. We say that the Schwarz function on the double is $S(z)$ if the point is on the front side, and $\bar{z}$ if the point belongs to the back side (here we understand $S(z)$ as a function defined on the complex curve, not just on the physical sheet).

The number of sheets of the curve is the number of poles (counted with their multiplicity) of the function $A(z)$ plus one. Indeed, poles of A are poles of the Schwarz function on the front side of the double. On the back side, there is also a pole at infinity. Since $S(z = \infty) = A(\infty)$, we have $\bar{S}(\bar{z} = \overline{A(\infty)}) = \infty$. Therefore, the factor $a(z)$ is a polynomial with zeros at the poles of $A(z)$ and at $\overline{A(\infty)}$, and

$$d \equiv \text{number of sheets} = \text{number of poles of } A + 1.$$

The front and back sides meet at planar curves $\bar{z} = S(z)$. These curves are boundaries of the droplets. We repeat that not all droplets are physical. Some of them may belong to unphysical sheets, figure 3.

Boundaries of droplets, physical and virtual, form a subset of the **a**-cycles on the curve. Their number cannot exceed the genus of the curve plus one,

$$\text{number of droplets} \leqslant g + 1.$$

The sheets meet along cuts located inside droplets. The cuts that belong to physical droplets show up on unphysical sheets. On the other hand, some cuts show up on the physical sheet (figure 3). They correspond to droplets situated on unphysical sheets.

The Riemann–Hurwitz theorem computes the genus of the curve as

$$g = \text{half the number of branching points} - d + 1.$$

With the help of the Stokes formula, the numbers $\{\nu_\alpha\}$ are identified with areas of the droplets: $|\nu_a| = \frac{1}{2\pi\hbar}\int_{D_a} \mathrm{d}^2 z$. For a nondegenerate curve, these numbers are not necessarily positive. Negative numbers correspond to droplets located on unphysical sheets. In this case,

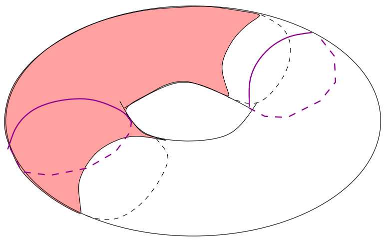

Figure 3. Physical and unphysical droplets on a torus. The physical sheet (shaded) meets the unphysical sheet along the cuts. The cut situated inside the unphysical droplet appears on the physical sheet. The boundaries of the droplets (physical and virtual) belong to different sheets. This torus is the Riemann surface corresponding to the ensemble with the potential $V(z) = -\alpha \log(1 - z/\beta) - \gamma z$.

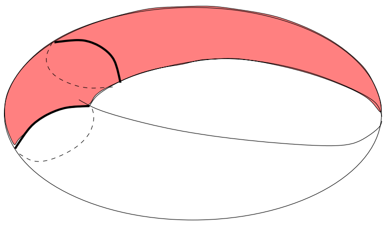

Figure 4. Degenerate torus corresponds to the algebraic domain for the Joukowsky map.

$\{\nu_a\}$ do not correspond to the number of eigenvalues located inside each droplet, as it is the case for algebraic domains, when all filling numbers are positive.

3.7.3. Degeneration of the spectral curve. Degeneration of the complex curve gives the most interesting physical aspects of growth. There are several levels of degeneration. We briefly discuss them below.

Algebraic domains and double points. A special case occurs when the Schwarz function on the physical sheet is meromorphic. It has no other singularities than poles of A. This is the case of algebraic domains. They appear in the semiclassical case. This situation occurs if cuts on the physical sheet, situated outside physical droplets, shrink to points, i.e., two or more branching points merge. Then the physical sheet meets other sheets along cuts situated inside physical droplets only and also at some points on their exterior (*double points*). In this case the Riemann surface degenerates. The genus is given by the number of physical droplets only. The filling factors are all positive.

In the case of algebraic domains, the physical branch of the Schwarz function is a well-defined meromorphic function. Analytic continuations of $\bar{z}$ from different disconnected parts of the boundary give the same result. In this case, the Schwarz function can be written through the Cauchy transform of the physical droplets,

$$S(z) = A(z) + \frac{1}{\pi} \int_D \frac{\mathrm{d}^2\zeta}{z - \zeta}. \tag{74}$$

Although algebraic domains occur in physical problems such as Laplacian growth, their semiclassical evolution is limited. Almost all algebraic domains will be broken in a growth process. Within a finite time (the area of the domain) they degenerate further into critical curves. The Gaussian potential (the Ginibre–Girko ensemble), which leads to a single droplet of the form of an ellipse is a known exception.

Critical degenerate curves. Algebraic domains appear as a result of merging of simple branching points on the physical sheet. The double points are located outside physical droplets. Remaining branching points belong to the interior of physical droplets. Initially, they survive in the degeneration process. However, as known in the theory of Laplacian growth, the process necessarily leads to a further degeneration. Sooner or later, at least one of the interior branching points merges with one of the double points in the exterior. Curves degenerated in this manner are called *critical*. For the genus one and three this degeneration is discussed below.

Since interior branching points can only merge with exterior branching points on the boundary of the droplet, the boundary develops a cusp, characterized by a pair p, q of mutually prime integers. In local coordinates around such a cusp, the curve looks like $x^p \sim y^q$. The fact that the growth of algebraic domains always leads to critical curves is known in the theory of Laplacian growth as finite-time singularities.

The degeneration process seems to be a feature of the semiclassical approximation. Curves treated beyond this approximation never degenerate.

3.7.4. Example: genus one curve. The potential is $V(z) = -\alpha\log(1 - z/\beta) - \gamma z$, $A(z) = -\frac{\alpha}{z-\beta} - \gamma$. There is one pole at $z = \beta$ on the first (physical) sheet. At $z = \infty$ on the first sheet $S(z) \to -\gamma + \frac{n\hbar-\alpha}{z}$. Therefore, the Schwarz function has another pole at the point $-\bar{\gamma}$ on another sheet. All the poles are simple. According to the general arguments of section 3.7.2, the number of sheets is 2, the number of branching points is 4. The genus is 1. The curve has the form

$$f(z,\bar{z}) = z^2\bar{z}^2 + k_1 z^2\bar{z} + \bar{k}_1 z\bar{z}^2 + k_2 z^2 + \bar{k}_2\bar{z}^2 + k_3 z\bar{z} + k_4 z + \bar{k}_4\bar{z} + h = 0.$$

The points at infinity and $-\bar{\gamma}$ belong to the second sheet of the algebraic covering. Summing up,

$$S(z) = \begin{cases} -\dfrac{\alpha}{z-\beta} & \text{as} \quad z \to \beta_1, \\ \left(-\gamma + \dfrac{n\hbar-\alpha}{z}\right) & \text{as} \quad z \to \infty_1, \\ \dfrac{n\hbar-\bar{\alpha}}{z+\bar{\gamma}} & \text{as} \quad z \to -\bar{\gamma}_2, \\ \left(\bar{\beta} - \dfrac{\bar{\alpha}}{z}\right) & \text{as} \quad z \to \infty_2, \end{cases}$$

where 1 and 2 indicate the sheets.

Poles and residues of the Schwarz function determine all the coefficients of the curve $f(z,\bar{z}) = a(z)(\bar{z} - S^{(1)}(z))(\bar{z} - S^{(2)}(z)) = \overline{a(z)}(z - \bar{S}^{(1)}(\bar{z}))(z - \bar{S}^{(2)}(\bar{z}))$ except one. The behavior at ∞ of $z, \bar{z}$ gives $k_1 = \gamma - \bar{\beta}$, $k_2 = -\gamma\bar{\beta}$. Hereafter we choose the origin by setting $\gamma = 0$. The equation of the curve then reads $f_n(z,\bar{z}) = 0$, where $f_n(z,\bar{z})$ is given by

$$z^2\bar{z}^2 - z^2\bar{z}\bar{\beta} - z\bar{z}^2\beta + (|\beta|^2 + \alpha + \bar{\alpha} - n\hbar)z\bar{z} + z\bar{\beta}(n\hbar - \alpha) + \bar{z}\beta(n\hbar - \bar{\alpha}) + h_n. \tag{75}$$

The free term h_n is to be determined by filling factors of the two droplets ν_1 and $\nu_2 = n - \nu_1$. A detailed analysis shows that the droplets belong to different sheets (figure 3). Therefore, ν_2 is negative.

A boundary of a physical droplet is given by the equation $\bar{z} = S^{(1)}(z)$ (figure 1). The second droplet belongs to the unphysical sheet. Its boundary is given by $\bar{z} = S^{(2)}(z)$. The explicit form of both branches is

$$S^{(1,2)} = \frac{1}{2}\bar{\beta} - \frac{\beta(n\hbar - \bar{\alpha}) + (\alpha + \bar{\alpha} - n\hbar)z \mp \sqrt{(z - z_1)(z - z_2)(z - z_3)(z - z_4)}}{2(z - \beta)z},$$

where the branching points z_i depend on h_n.

If the filling factor of the physical droplet is equal to n, the cut inside the unphysical droplet is of the order of $\sqrt{\hbar}$. Although it never vanishes, it shrinks to a double point $z_3 = z_4 = z_*$ in a semiclassical limit. The sheets meet at the double point z_* rather than along the cut: $\sqrt{(z - z_1)(z - z_2)(z - z_3)(z - z_4)} \to (z - z_*)\sqrt{(z - z_1)(z - z_2)}$. In this case, genus of the curve reduces to zero and the exterior of the physical droplet becomes an algebraic domain. This condition determines h, and also the position of the double point (figure 4). The double point is a saddle point for the level curves of $f(z, \bar{z})$. If all the parameters are real, the double point is stable in the x-direction and unstable in the y-direction.

If this solution is chosen, the exterior of the physical droplet can be mapped to the exterior of the unit disk by the Joukowsky map

$$z(w) = rw + u_0 + \frac{u}{w - a}, \qquad |w| > 1, \quad |a| < 1. \tag{76}$$

The inverse map is given by the branch $w_1(z)$ (such that $w_1 \to \infty$ as $z \to \infty$) of the double-valued function

$$w_{1,2}(z) = \frac{1}{2r}\left[z - u_0 + ar \pm \sqrt{(z - z_1)(z - z_2)}\right], \qquad z_{1,2} = u_0 + ar \mp 2\sqrt{r(u + au_0)}.$$

The function

$$\bar{z}(w^{-1}) = rw^{-1} + \bar{u}_0 + \frac{\bar{u}}{w^{-1} - \bar{a}} \tag{77}$$

is a meromorphic function of w with two simple poles at $w = 0$ and $w = \bar{a}^{-1}$. Treated as a function of z, it covers the z-plane twice. Two branches of the Schwarz function are $S^{(1,2)}(z) = \bar{z}\left(w_{1,2}^{-1}(z)\right)$. On the physical sheet, $S^{(1)}(z) = \bar{z}(w_1(z))$ is the analytic continuation of $\bar{z}$ away from the boundary. This function is meromorphic outside the droplet. Apart from a cut between the branching points $z_{1,2}$, the sheets also meet at the double point $z_* = -\bar{\gamma} + a^{-1}r\,e^{2i\phi}$, where $S^{(1)}(z_*) = S^{(2)}(z_*)$, $\phi = \arg\left(ar + \frac{u\bar{a}}{1-|a|^2}\right)$.

Analyzing singularities of the Schwarz function, one connects parameters of the conformal map with the deformation parameters:

$$\begin{cases} \gamma = \dfrac{\bar{u}}{\bar{a}} - \bar{u}_0, \\ n\hbar - \bar{\alpha} = r^2 - \dfrac{ur}{a^2}, \\ \beta = \dfrac{r}{\bar{a}} + u_0 + \dfrac{u\bar{a}}{1 - |a|^2} \end{cases} \tag{78}$$

$$\text{Area of the droplet} \sim n\hbar = r^2 - \frac{|u|^2}{(1 - |a|^2)^2}.$$

A critical degeneration occurs when the double point merges with a branching point located inside the droplet ($z_* = z_2$) to form a triple point z_{**}. This may happen on the boundary only. At this point, the boundary has a $(2, 3)$ cusp. In local coordinates, it is $x^2 \sim y^3$. This is a critical point of the conformal map: $w'(z_{**}) = \infty$. A critical point inevitably results from the evolution at some finite critical area.

A direct way to obtain the complex curve from the conformal map is the following. First, rewrite (76) and (77) as

$$\begin{cases} z - u_0 + ar = rw + a(z+\bar\gamma)w^{-1} \\ \bar z - \bar u_0 + \bar a r = rw^{-1} + \bar a(\bar z + \gamma)w, \end{cases} \tag{79}$$

and treat w and $1/w$ as independent variables. Then impose the condition $w \cdot w^{-1} = 1$. One obtains

$$\left|\det\begin{bmatrix} z - u_0 + ar & a(z+\gamma) \\ \bar z - \bar u_0 + \bar a r & r \end{bmatrix}\right|^2 = \left(\det\begin{bmatrix} r & a(z+\bar\gamma) \\ \bar a(\bar z+\gamma) & r\end{bmatrix}\right)^2.$$

This gives the equation of the curve and in particular h, in terms of u, u_0, r, a and eventually through the deformation parameters α, β, γ and t.

The semiclassical analysis gives a guidance for the form of the recurrence relations. Let us use an ansatz for the L-operator, which resembles the conformal map (76),

$$L = r_n\hat w + u_n^{(0)} + (\hat w - a_n)^{-1}u_n,$$

so that

$$(\hat w - a_n)L = (\hat w - a_n)r_n\hat w + (\hat w - a_n)u_n^{(0)} + u_n, \tag{80}$$

$$L^\dagger(\hat w^{-1} - \bar a_n) = \hat w^{-1}r_n(\hat w^{-1} - \bar a_n) + \bar u_n^{(0)}(\hat w^{-1} - \bar a_n) + \bar u_n, \tag{81}$$

where $\hat w$ is the shift operator $n \to n+1$.

Now we follow the procedure of the previous section. Since the potential has only one pole, $\mathcal{L}_n$ can be cast into 2×2 matrix form. Let us apply the lines (80), (81) to an eigenvector (c_n, c_{n+1}) of a yet unknown operator $\mathcal{L}_n$, and set the eigenvalue to be $\tilde z$,

$$\begin{cases} \left(z + r_{n-1}a_{n-1} - u_n^{(0)}\right)c_n = r_nc_{n+1} + a_{n-1}(z+\bar\gamma_{n-1})c_{n-1} \\ \left(\tilde z + r_n\bar a_n - \bar u_{n+1}^{(0)}\right)c_n = \bar a_{n+1}(\tilde z + \gamma_{n+1})c_{n+1} + r_nc_{n-1}. \end{cases} \tag{82}$$

We have defined $\bar\gamma_n = \frac{u_n}{a_n} - u_n^0$. The equations are compatible if c_{n-1} and c_{n+1} found through c_n differ by the shift $n \to n+2$. We have

$$c_{n+1} = \frac{c_n}{d_n}\det\begin{vmatrix} z + r_{n-1}a_{n-1} - u_n^{(0)} & a_{n-1}(z+\bar\gamma_{n-1}) \\ \tilde z + r_n\bar a_n - \bar u_{n+1}^{(0)} & r_n\end{vmatrix} = c_n\frac{\widetilde{\mathcal{D}}_n}{d_n}, \tag{83}$$

$$c_{n-1} = \frac{c_n}{d_n}\det\begin{vmatrix} r_n & z + r_{n-1}a_{n-1} - u_n^{(0)} \\ \bar a_{n+1}(\tilde z + \gamma_{n+1}) & \tilde z + r_n\bar a_n - \bar u_{n+1}^{(0)}\end{vmatrix} = c_n\frac{\mathcal{D}_n}{d_n}, \tag{84}$$

where

$$d_n = \det\begin{vmatrix} r_n & a_{n-1}(z+\bar\gamma_{n-1}) \\ \bar a_{n+1}(\tilde z + \gamma_{n+1}) & r_n\end{vmatrix}. \tag{85}$$

This yields the curve

$$\widetilde{\mathcal{D}}_n \cdot \mathcal{D}_{n+1} = d_nd_{n+1}. \tag{86}$$

Comparing the two forms of the curve (75) and (86), we obtain the conservation laws of growth

$$\gamma = \gamma_n = \frac{\bar u_n}{\bar a_n} - \bar u_n^0, \tag{87}$$

$$\beta = \frac{r_n}{\bar{a}_{n+1}} + u_{n+1}^{(0)} + \frac{u_{n+1}^{(0)} a_n \bar{a}_{n+1}}{1 - a_n \bar{a}_{n+1}}, \tag{88}$$

$$n\hbar - \bar{\alpha} = r_n r_{n+1} - \frac{r_{n+1} u_{n+1}}{a_n a_{n+1}}. \tag{89}$$

They are the quantum version of (78).

3.8. Continuum limit and conformal maps

The geometrical meaning of the complex curve (67) is straightforward: at fixed shape parameters t_k and area parameter $\hbar$, increasing n yields growing domains that represent the support of the corresponding $n \times n$ model. A remarkable feature of this process is that it preserves the external harmonic moments of the domain $\mathbb{D}_n$,

$$t_k(n) = t_k(n-1), \qquad t_k(n) = -\frac{1}{\pi k} \int_{\mathbb{C}\setminus\mathbb{D}_n} \frac{\mathrm{d}^2 z}{z^k}, \qquad k \geqslant 1. \tag{90}$$

The only harmonic moment which changes in this process is the normalized area $t_0 = \frac{1}{\pi} \int \mathrm{d}^2 z$, and it increases in increments of $\hbar$ (hence the meaning of $\hbar$ as quantum of area). We may say that the growth of the NRM ensemble consists of increasing the area of the domain by multiples of $\hbar$, while preserving all the other external harmonic moments. The continuum version of this process, known as *Laplacian growth*, is a famous problem of complex analysis. It arises in the two-dimensional hydrodynamics of two non-mixing fluids, one inviscid and the other viscous, upon neglecting the effects of surface tension, where it is known as the Hele–Shaw problem. The following sections discuss this classical problem in great detail.

As we will see, Laplacian growth can be restated simply as a problem of finding the uniform equilibrium measure, subject to constraints on the total mass, and the asymptotic expansion of the logarithmic potential at infinity. As long as a classical solution exists, the machinery of NRM does not seem necessary. However, Laplacian growth (as a class of processes) is characterized by finite-time singularities. In that case, the only way to reformulate the problem is similar to the Saff–Totik approach to the extremal measure, and is deeply related to weighted limits of orthogonal polynomials in the complex plane.

4. Laplacian growth

4.1. Introduction

Laplacian growth (LG) is defined as the motion of a planar domain, whose boundary velocity is a gradient of the Green function of the same domain (also called a *harmonic measure*). This deceivingly simple process appears to be connected to an impressive number of non-trivial physical and mathematical problems [49, 50]. As a highly unstable, dissipative, non-equilibrium and nonlinear phenomenon, it is famous for producing different universal patterns [51, 52].

Numerous non-equilibrium physical processes of apparently different nature are examples of Laplacian growth: viscous fingering [51], slow freezing of fluids (Stefan problem) [53], growth of snowflakes [54], crystal growth, amorphous solidification [55], electrodeposition [56], bacterial colony growth [57], diffusion-limited aggregation (DLA) [58], motion of a charged surface in liquid Helium [59] and secondary petroleum production [60], to name just a few.

A major consequence from the current development of the subject is a discovery of a new and unexpectedly fruitful mathematical structure, which is capable to predict and explain key physical observations in regimes, totally inaccessible by any other available mathematical method.

The first section of this section is a brief history of physics covered by the Laplacian growth. The second section addresses in detail the exact time-dependent solutions of the Laplacian growth equation, and the last is a detailed presentation of the analytic and algebraic-geometric structure of Laplacian growth.

4.2. Physical background

Darcy's law. In 1856, while completing a hydrological study for the city of Dijon, H Darcy noted that the rate of flow (volume per unit time) Q through a given cross-section, t_0, is (a) proportional to t_0, (b) inversely proportional to the length, L, taken between positions of efflux and influx and (c) linearly proportional to pressure difference, Δp, taken between the same two levels. In short,

$$Q = -\frac{kt_0}{L}\Delta p, \tag{91}$$

where k is a positive constant. As one can see, Darcy's observation coincides with Ohm's law, upon identifying Q, Δp and k as the total current through the cross-section t_0, the electric potential difference, and the electrical conductivity, respectively. Rewriting (91) in a differential form, as for Ohm's law, we obtain

$$\mathbf{v} = -k\nabla p, \tag{92}$$

where $\mathbf{v}$ is the velocity vector field of fluid particles, properly coarse grained to assure its smoothness over infinitesimally small volumes. Here the kinetic coefficient, k (the same as in (92)), is called a (hydraulic) conductivity and can depend on position. Equation (92) constitutes Darcy's law in a differential form. For homogenous k,

$$\mathbf{v} = \nabla(-kp), \tag{93}$$

Darcy's law merely states that a flow through uniform porous media (sand in Darcy's experiments) is purely potential (no vortices), where the pressure field, p, is a velocity potential up to a constant factor. Assuming constant k and the fluid incompressible, $\nabla \cdot \mathbf{v} = 0$, we find that pressure p is a harmonic function,

$$\nabla^2 p = 0. \tag{94}$$

As seen from purely dimensional considerations, the conductivity k equals

$$k = C\frac{d^2}{\mu}, \tag{95}$$

where d is the average linear size of a pore in cross-section, μ is the dynamical viscosity of the fluid under consideration and the dimensionless coefficient, C, is usually small and media-dependent. (It is of the order of the density of voids in a given porous medium.)

It follows from (92) and (95), that if μ is negligibly small (an almost inviscid liquid), pressure gradients are also negligibly small, regardless of how fast fluid moves (but still much slower than the velocity of sound in this liquid in order to assure incompressibility assumed earlier).

Laplacian growth in porous media. Assume that a fluid with a viscosity μ_1 occupying a domain $D_1(t)$ at the moment t pushes another fluid with a viscosity μ_2 occupying the domain $D_2(t)$ at the same time t through a uniform porous media. Then the Laplace equation will hold for both pressures p_1 and p_2 corresponding to domains D_1 and D_2 respectively,

$$\nabla^2 p_i = 0 \qquad \text{in} \quad D_i(t), \tag{96}$$

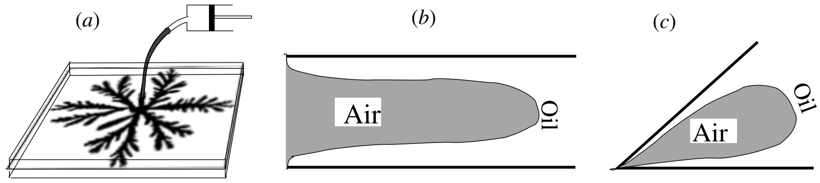

Figure 5. Laplacian growth in a Hele–Shaw cell for the radial (*a*), channel (*b*) and wedge (*c*) geometries.

where $i = 1, 2$. At the interface $\Gamma(t)$, where two fluids meet (but do not mix), their normal velocities coincide because of continuity and equal to the normal component V_n of the velocity of the boundary, $\Gamma(t) = \partial D_1 = -\partial D_2$,

$$\mathbf{v_1}|_n = \mathbf{v_2}|_n = V_n \qquad \text{at} \quad \Gamma(\mathrm{t}). \tag{97}$$

The pressure field p at the interface $\Gamma(t)$ (by the Laplace law) has a jump equal to the mean local curvature κ multiplied by the *surface tension* σ,

$$p_1 - p_2 = \sigma\kappa \qquad \text{at} \quad \Gamma(\mathrm{t}). \tag{98}$$

Unless the local curvature is very high, this surface tension correction is usually very small, and so is often neglected. If to supplement the last three equations by boundary conditions at external walls or/and at infinity (they may include sources/sinks of fluids either extended or point-like), then the free boundary problem of finding $\Gamma(t)$ by initially given D_1 and D_2 is completely formulated.

The process described by (96)–(98) is typical for various geophysical systems, for instance for petroleum production, where a less viscous fluid (usually water) pushes a much more viscous one (oil) toward production wells. This process is very unstable and most initially smooth water/oil fronts will quickly break down and become fragmented.

The Hele–Shaw cell. In 1898, H S Hele–Shaw proposed an interesting way to observe and study two-dimensional fluid flows by using two closely-placed parallel glass plates with a gap between them occupied by the fluid under consideration [61]. This simple device appears to be very useful in various investigations and is now called a Hele–Shaw cell after its inventor. Remarkably, a viscous fluid, governed in 3D by the Stokes law,

$$\mu\nabla^2\mathbf{v} = \nabla p, \tag{99}$$

after being trapped in a gap of a width b, between the plates of a Hele–Shaw cell, obeys Darcy's law (92) with a conductivity equal to $k = b^2/(12\mu)$. The derivation of the formula

$$\mathbf{v} = -\frac{b^2}{12\mu}\nabla p, \tag{100}$$

which is to be understood as a 2D vector field in a plane parallel to the Hele–Shaw cell plates, is rather trivial and results from the averaging of (99) over the dimension perpendicular to the plates [62, 63]. Thus, displacement of viscous fluid by the (almost) inviscid one in a Hele–Shaw cell became a major experimental tool to investigate a 2D Laplacian growth. Various versions of 2D Laplacian growth in a Hele–Shaw cell corresponding to different geometries are shown in figure 5.

Idealized Laplacian growth. In 1945, Polubarinova–Kochina [64] and Galin [65] simultaneously, but independently, derived a nonlinear integro-differential equation for an oil/water interface in 2D Laplacian growth, after neglecting surface tension, σ, and water viscosity, $\mu_{\text{water}} = 0$. Assuming for simplicity a singly connected oil bubble, occupying a domain $D(t)$ surrounded by water and having a sink at the origin, $0 \in D(t)$, we will obtain this equation starting from the system

$$\begin{cases} \nabla^2 p = \rho & \text{in } D(t), \\ p = 0 & \text{at the interface, } \Gamma(t) = \partial D(t) \\ V_n = -\partial_n p & \text{at the interface, } \Gamma(t), \end{cases} \tag{101}$$

where ρ and ∂_n are density of sources and the normal derivative respectively. This system is a reduction of (96), (97) after simplifications mentioned above and using the fact that the normal boundary velocity, V_n, equals to the normal components of the fluid velocity at the boundary, which is $-\partial_n p$ by virtue of the Darcy law (92). Here and below the conductivity k is scaled to one. The density of sources, ρ, in this case equals $\rho(z) = -\delta^2(z)$, which corresponds to a sink of unit strength located at the origin.

The Laplacian growth equation. Coming back to the derivation, we apply the conformal map from the unit disc in the complex plane $w = \exp(-p + \mathrm{i}\phi)$, where the (stream) function $\phi(x, y)$ is harmonically conjugate to $p(x, y)$, into the domain $D(t)$ in the 'physical' complex plane $z = x + \mathrm{i}y$, and zero maps to zero with a positive coefficient. Denoting the moving boundary as $z(t, l)$, where l is the arclength along the interface, one obtains

$$V_n = \mathrm{Im}(\bar{z}_t z_l) = -\partial_n p = \partial_l \phi. \tag{102}$$

It is trivial to see that the chain of three equalities in (102) represent, respectively, the definition of V_n in terms of a moving complex boundary, $z(t, l)$ (the first equality), the kinematic identity expressed by the last equation in the system (101) (the second one), and the Cauchy–Riemann relation (the last one) between p and ϕ. After reparametrization, $l \to \phi$, we arrive to the equation

$$\mathrm{Im}(\bar{z}_t z_\phi) = 1, \tag{103}$$

which possesses many remarkable properties, as will be seen below. Equation (103) is usually referred as the Laplacian growth equation (LGE) or the Polubarinova–Galin equation. In [64, 65] it was noted a fully unexpected feature of equation (103): the boundary, $z(t, \phi)$, taken initially as a polynomial of $w = \exp(\mathrm{i}\phi)$, will remain a polynomial of the same degree with time-dependent coefficients during the course of evolution, so new degrees of freedom, describing the moving boundary, will not appear.

An even more remarkable observation concerning equation (103), belongs to Kufarev [66], who found that a boundary taken as a rational function with respect to $w = \exp(\mathrm{i}\phi)$ will stay as such during the evolution. Moreover, he managed to integrate this dynamical system explicitly, and found first integrals of motion associated with moving poles and residues of the conformal map, $z(t, \exp(\mathrm{i}\phi))$, describing the boundary. The authors [64–66] have however noted that all the solutions obtained are short lived, both because of instability and due to the finite volume of $D(t)$, which is destined to shrink, because of a sink(s) located inside. We will address these interesting observations in detail in the second subsection of this section.

LGE in the evolutionary form. It is of help to present (103) in the evolutionary form, defined as the dynamical system, where the time derivative constitutes the LHS and does enter the RHS. For this purpose we rewrite (103) as

$$\bar{z}_t z_\phi = \mathrm{i} + t_0,$$

where t_0 is real. Dividing both sides by $|z_\phi|^2$, we will obtain

$$\frac{\bar{z}_t}{\bar{z}_\phi} = \frac{\mathrm{i} + t_0}{|z_\phi|^2}.$$

Taking the conjugates from both sides and multiplying by i, we will have

$$\mathrm{i}\frac{z_t}{z_\phi} = \frac{1+\mathrm{i}t_0}{|z_\phi|^2}.$$

The LHS is the analytic function outside the unit disk in the w-plane. In accordance with the last equation, the real part of this analytic function along the unit circumference equals $|z_\phi|^{-2}$. To recover the analytic function from the boundary value of its real part at the unit circle is a well-known procedure involving either the Hilbert transform or the Schwarz integral. The result is

$$\mathrm{i}z_t = -z_\phi \int_0^{2\pi} \frac{\mathrm{e}^{\mathrm{i}s} + \mathrm{e}^{\mathrm{i}\phi}}{\mathrm{e}^{\mathrm{i}s} - (1+\epsilon)\,\mathrm{e}^{\mathrm{i}\phi}} \frac{1}{|z_s|^2} \frac{\mathrm{d}s}{2\pi}, \tag{104}$$

where an infinitesimally small positive ϵ indicates correct limiting value of the integral while approaching the unit circumference. This useful formula was obtained by Shraiman and Bensimon in 1984 [67]. This expression for (103) in the evolutionary form reveals the nonlocal nature of Laplacian growth due to the integral in the RHS.

Equation (104) helps to prove a beautiful statement that every singularity of the function $z(t, w)$ moves toward the unit circle from inside, or in other words the radial component of the 2D velocity of any singularity of the conformal map is positive. To prove the claim, we replace ϕ in (104) by W, defined earlier as $W = -p + \mathrm{i}\phi$. Then after we note that

$$-\frac{z_t(t, e^W)}{z_W(t, e^W)} = \left[\frac{\mathrm{d}W}{\mathrm{d}t}\right]_{z=\mathrm{const}}$$

and that near a singular point $w = a$ we can replace $W = \log(w) = \log(a)$, we can rewrite the real part of (104) as

$$\frac{\mathrm{d}\log|a|}{\mathrm{d}t} = \left[\frac{1}{|z_w|^2}\right]_{w=a} > 0. \tag{105}$$

Thus, we proved that each singular point of the conformal map moves toward the unit circle from inside, so the origin is a repellor for this dynamical system, and the unit circumference is an attractor.

Diffusion limited aggregation. The physics section of the survey cannot be completed without mentioning a fascinating discovery by T A Witten and L M Sander, who observed [58] in 1981 that a cluster on a 2D square lattice, grown by subsequent attaching to it a Brownian diffusive particles, eventually becomes a self-similar fractal (see figure 6) with a robust universal fractal (Haussdorff) dimension given by

$$D_0 = \lim_{\epsilon\to 0} \frac{\log N(\epsilon)}{\log(1/\epsilon)} = 1.71 \pm 0.01, \tag{106}$$

where $1/\epsilon$ is a linear size of a cluster measured by a small 'yard stick', ϵ, and $N(\epsilon)$ is a minimal number of (small) boxes with a side ϵ, which covers the cluster.

Remarkably, this fractal appeared to be self-similar after appropriate statistical averaging. This means that its higher multi-fractal dimensions, D_q, defined as

$$D_q = \lim_{\epsilon\to 0} \frac{\log\left(\sum_i^N p_i^q\right)}{\log(1/\epsilon)}, \tag{107}$$

where p_i stands for a portion of a tiny box of size ϵ, covered by the cluster under consideration, appear to be equal to each other, and to D_0, which is 1.71, as indicated above. Later, these findings were significantly clarified and refined in many respects, but the major challenge: how

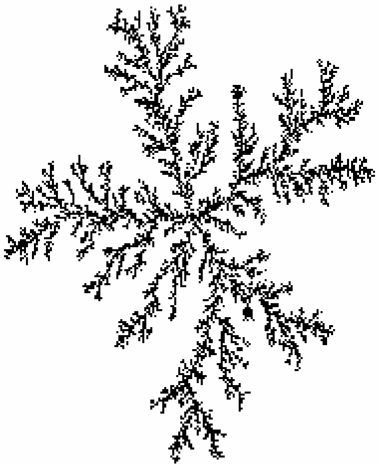

Figure 6. A DLA cluster, $n = 100\,000$.

to calculate the universal dimension defined above still is an open question (see a relatively recent review [68] and references therein).

This problem is tightly connected to the Laplacian growth. Until very recently there were numerous claims that the DLA process is drastically different from the Laplacian growth, and even statements appeared that the DLA and fractals grown in Laplacian growth belong to different universality classes [69]. However, the recent experiments by Praud and Swinney [70] made crystal clear that the multi-fractal spectrum of a cluster grown in a viscous fingering process in a Hele–Shaw cell (that is a Laplacian growth) coincides with the DLA spectrum up to the margin accuracy of 1%, which is the maximal accuracy in these measurements. Thus, despite of its discrete and a stochastic nature, DLA can be understood by a continuum and deterministic Laplacian growth (101).

Related problems. Below is a list of physical problems connected with Laplacian growth.

First, there is the so-called singular Laplacian growth, where a growing domain consists of needles with zero areas and divergent curvature at the tip. The mathematical description for this dynamics should be reformulated, since the gradient of pressure p diverges near moving needle tips, so the boundary velocity should be replaced by an appropriately regularized law. Interesting works by Derrida and Hakim [71], and by Peterson [72] in this direction deserve special attention.

There is also a considerable amount of works in so-called nonlinear mean-field dynamics, where a phase field is involved, which gradually changes from unity in one of moving phases toward zero inside the second one [73, 74]. Many of these processes, including dynamics of microstructure [75] in materials, growth of bacterial colonies in nutritional environment [76], and spinodal decomposition [77], governed by the time-dependent Ginzburg–Landau and the Cahn–Hilliard equations, are reduced to the Laplacian growth interface dynamics in a special singular limit, when the phase field degenerates to a step-function, thus becoming a characteristic function of a moving domain with a well-defined boundary [78, 79]. This is certainly worth to mention, both because it significantly enriches a physical process by introducing an additional field (the phase field) and since this is conceptually related to a random matrix approach to Laplacian growth, addressed in the survey, and where a distribution of eigenvalue support will play a role of a mean-field phase, introduced in this paragraph.

Let us also mention several more 'selection puzzles', which belong to the Laplacian growth in various settings: selection of a shape of a separated inviscid bubble, observed by Taylor and Saffman in a viscous flow in a rectangular Hele–Shaw cell [80] from a continuous family of possible solutions (not to be confused with the Saffman–Taylor fingers family described in [51]); selection of a so-called skinny finger in a Hele–Shaw cell accelerated by a tiny inviscid bubble near the nose of a finger [81]; and prediction of the periodicity for the so-called side-branching structure in dendritic growth [82]. These phenomena have the same (or almost the same) mathematical description.

Another important comment about physics of Laplacian growth is that Darcy's law (92) is invalid near walls of a Hele–Shaw cell, including proximity to both parallel plates. This is because averaging of the Stokes flow, $\mu\nabla^2\mathbf{v} = \nabla p$, given by (99) will no longer bring us to (100), due to boundary layer effects. This apparent difficulty gives rise to the study of an interface dynamics with a Stokes flow, which is an extension of the Hele–Shaw (Darcy's) flow. The Stokes flow also contains remarkable physics and beautiful mathematics [83–85], which is still yet to be fully understood.

4.3. Exact solutions

Cardioid. Consider the equation of motion for the droplet boundary under Laplacian growth

$$\mathrm{Im}(\bar{z}_t z_\phi) = Q, \tag{108}$$

where $2\pi Q$ stands for a rate of a source (sink). Here, $z(t, \mathrm{e}^{\mathrm{i}\phi})$ is conformal inside the unit circle, $|w| < 1, 0 \to 0$, and $w = \mathrm{e}^{\mathrm{i}\phi}$ in the equation. When one tries to solve (108), the solution

$$z = r(t)\,\mathrm{e}^{\mathrm{i}\phi} \tag{109}$$

comes to mind first, as the simplest one. It describes initially circular droplet centered at the origin, which uniformly grows (shrinks) while continuing to be a circle. Indeed, substituting (109) into (108) one obtains

$$r(t) = \sqrt{2(|Q|T + Qt)}, \tag{110}$$

where a constant of integration T stands for an initial time. When $Q < 0$ (suction), the circle shrinks to a point at $t = T$, and the solution (109) ceases to exist after T. Could one find any other exact solutions, less trivial than given by (109)?

Remarkably, the answer is yes, despite of nonlinearity of the Laplacian growth equation (108). Let us add to (109) an initially small quadratic correction,

$$z = r(t)\,\mathrm{e}^{\mathrm{i}\phi} + a(t)\,\mathrm{e}^{2\mathrm{i}\phi}. \tag{111}$$

The domain bounded by the curve described by (111), named *a cardiod*, is connected if $|a| < r/2$. Substituting (111) into (108) one obtains two coupled nonlinear first-order ODEs w.r.t. r and a,

$$\begin{cases} r\dot{r} + 2a\dot{a} = Q \\ \dot{a}r + 2a\dot{r} = 0, \end{cases} \tag{112}$$

with an easily found solution

$$\begin{cases} ar^2 = a_0 \\ r^2 + 2a^2 = 2(|Q|t_0 + Qt) \end{cases} \tag{113}$$

with a_0 and t_0 as constants of integration. If $Q > 0$ (injection), the cardiod will grow becoming more and more like a circle during the evolution. If instead $Q < 0$ (suction) the cardioid (111) shrinks, deforms and ceases to exist after $t^* = t_0 + 3a_0^{2/3}\big/(\sqrt[3]{16}Q)$. This happens when the critical point of the conformal map given by (111) reaches the unit circle from outside. Then the cardioid ceases to be analytic and earns a needle-like cusp (a point of return with infinite curvature). This cusp is called type 3/2 (alternatively (2,3)-cusp) because in local Cartesian coordinates it is described by the equation $y^2 \sim x^3$. We will see later that this kind of cusps is typical for those solutions of Laplacian growth which cease to exist in finite time.

Polynomials. As a generalization, we are going to prove now that all polynomials of w, which describe boundaries of analytic domains when $|w| = 1$, are solutions of (108). Assume a droplet is initially described by a trigonometric polynomial (with all critical points lying outside the unit disk, because its interior conformally maps onto a droplet),

$$z = \sum_{k=1}^{N} a_k\, \mathrm{e}^{\mathrm{i}k\phi}. \tag{114}$$

Substituting (114) into (108), one obtains N coupled ODEs for time-dependent coefficients a_k, and remarkably there are no other degrees of freedom which appear during the evolution. In other words, the evolving droplet will continue to be described by the polynomial (114), with coefficients, a_k, changing in time in accordance with these ODEs,

$$\sum_{k=1}^{N-n}[ka_k\dot{\bar{a}}_{k+n} + (k+n)\dot{a}_k\bar{a}_{k+n}] = Q\delta_{n,0}, \qquad n = 0, 1, \ldots, N-1. \tag{115}$$

Moreover, (115) can be integrated explicitly. Indeed, we note first that the equation for $k = N-1$, namely

$$a_1\dot{\bar{a}}_N + N\dot{a}_1\bar{a}_N = 0 \tag{116}$$

is trivially solved with the answer

$$\bar{a}_N a_1^N = C_N, \tag{117}$$

where C_N is the constant of integration. Substituting (117) into the $(N-2)$ ndequation, which has a form

$$a_1\dot{\bar{a}}_{N-1} + (N-1)\dot{a}_1\bar{a}_{N-1} + a_2\dot{\bar{a}}_N + N\dot{a}_2\bar{a}_N = 0, \tag{118}$$

we note that the LHS of (118) is proportional to a full derivative from the expression

$$a_1^{N-1}\bar{a}_{N-1} + NC_N\frac{a_2}{a_1^2}$$

and is zero in accordance with the RHS of (118). Thus we obtain

$$a_1^{N+1}\bar{a}_{N-1} + NC_Na_2 = C_{N-1}a_1^2, \tag{119}$$

where C_{N-1} is a constant of integration. Knowing a_{N-1} and a_N in terms of a_1 and a_2 we can easily integrate the third equation from the end of the system (115), namely the $(N-3)$ rdequation. The result is

$$a_1^{N+2}\bar{a}_{N-2} + (N-1)C_{N-1}a_2a_1^2 + NC_Na_3a_1 - N(N+1)\frac{C_Na_2^2}{2} = C_{N-2}a_1^4. \tag{120}$$

Continuing in this way, we obtain an explicit dependence of $\bar{a}_k$ as a linear combination of constants of motion, C_k, with coefficients which are polynomial forms w.r.t. $a_1, a_2, \ldots$. The equation for $n = 0$ from (115) already constitutes the full derivative and, as such, is trivially integrated,

$$\sum_{k=1}^{N} k|a_k|^2 = 2(C_0 + Qt), \tag{121}$$

where C_0 is a constant of integration. Here the LHS is a (scaled) area of the droplet, and the equation states that the area changes linearly in time. In other words, we integrated the system (115), and the solutions are polynomial forms with respect to a_k, linear w.r.t. integrals of motion, C_k, explicitly obtained.

As in the case of cardioid, in the case $Q > 0$ the dynamics is stable and the droplet becomes eventually more and more round since all a_k decay in time, but a_1 in contrary, grows, as one can easily verify by looking to the system (115). If $Q < 0$, then the droplet shrinks and the solution ceases to exist in finite time. This happens because a critical point(s) hits a unit circle from outside manifesting a break of analyticity by making a cusp (of a $3/2$ kind in general case). Except such rare cases as a circle centered at the location of sink, the solution stops to exist prior to the formation of a cusp, because of the droplet being completely sucked by the sink.

The fact that a finite-time singularity (a cusp) is generic follows directly from (117): since the conformal radius, a_1, should decrease as the area shrinks, then the coefficient, a_N, grows in time by virtue of (117), eventually bringing the system to a cusp.

Now consider the external Laplacian growth, where an inviscid bubble, surrounded by a viscous fluid grows (shrinks) because of a source (sink) at infinity. Then we map conformally the exterior of the unit disk in the w-plane to the exterior of a bubble (viscous region) in the physical z-plane with a simple pole and positive residue (which is a conformal radius) at infinity.

Here an analogy of the polynomial ansatz (115) will be the formula

$$z = \sum_{k=-1}^{N} a_k \, \mathrm{e}^{-\mathrm{i}k\phi}, \tag{122}$$

where $a_{-1} = r$ is the conformal radius, that is the radius of a circle perturbed by the rest of a_k's. This case is also integrable in a way, very similar to the interior case shown above [86]. One can also see that for an unstable LG, that is a growing bubble in the exterior problem, a finite time cusp is unavoidable. Indeed, the system of ODEs for the ansatz (122) will look the same as (115), but with values of n extended from -1 to $N + 1$. Thus one can easily see that $a_N = C_N r^{N-1}$. This means that the highest harmonic will grow faster than a conformal radius, which should eventually break domain's analyticity through a cusp [87].

The area, t_0, of the growing bubble in this case equals

$$t_0 = |r|^2 - \sum_{k>0} k|a_k|^2. \tag{123}$$

Consider the simplest non-trivial example for (122), which describes a shape with three-fold symmetry,

$$z(w) = rw + \frac{a}{w^2}, \tag{124}$$

we have

$$a = 3Cr^2, \tag{125}$$

where C is a constant of integration, and the scaled area of the droplet identified with time t in this case is

$$t_0 = t = |r|^2 - 18|C|^2|r|^4. \tag{126}$$

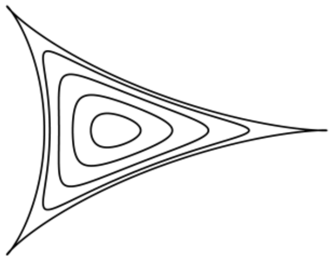

Figure 7. A Hele–Shaw droplet approaching the critical area.

Clearly, this polynomial in $|r|^2$ has a global maximum at r_c solving $36|r_c|^2|C|^2 = 1$. We call the corresponding value of the area $t_c = |r_c|^2/2$ *critical*, and conclude that the dynamics will lead to finite-time singularities for any initial condition $t_3 \neq 0$ figure 7.

In summary, we have shown that the Laplacian growth is integrable for polynomial (time-dependent) conformal mappings both in interior and exterior problem, and in both cases (growing of a bubble in the exterior and suction of a droplet for the interior problem) finite-time singularities in the form of the 3/2-cusps are unavoidable. This is caused by an ill-posedness of the Laplacian growth without regularized factors, such as surface tension.

Rational functions. Kufarev [66] found a class of rational solutions of equation (108) with simple poles. Let us show that all rational conformal maps from exterior (interior) of the unit disk to the exterior (interior) of a domain D are solutions of the LGE (108). We will include in our proof multiple poles for the sake of generality. Specifically, we claim that the expression

$$z = rw + \sum_{k=1}^{N} \sum_{l=1}^{P_k} \frac{A_{kl}}{(w - a_k)^{p_{kl}}}, \tag{127}$$

where $p_{kl} \in \mathbb{N}$, solves (108). Indeed, after substitution of (127) into (108), putting $w = \mathrm{e}^{\mathrm{i}\phi}$, we obtain a double sum, which we can decompose to elementary fractions with respect to $(\mathrm{e}^{\mathrm{i}\phi} - a_k)^{p_{kl}}$ by using repeatedly the identity

$$\frac{1}{(\mathrm{e}^{\mathrm{i}\phi} - a_k)(\mathrm{e}^{-\mathrm{i}\phi} - \bar{a}_l)} = \frac{1}{1 - a_k\bar{a}_l}\left(1 + \frac{a_k}{\mathrm{e}^{\mathrm{i}\phi} - a_k} - \frac{\bar{a}_l}{\mathrm{e}^{-\mathrm{i}\phi} - \bar{a}_l}\right). \tag{128}$$

Equating coefficients prior to all independent modes to zero and sum of all constants (the zeroth mode) to Q in accordance with (108), we see, after some algebra, that all the expressions are full derivatives, and after integration we obtain the following equations:

$$\begin{cases} r^2 - \sum_{k=1}^{N}\sum_{l=1}^{P_k} A_{kl}[\bar{z}(1/\bar{a}_k)]^{(p_{kl}-1)}/(p_{kl}-1)! = C + Qt \\ z(1/\bar{a}_k) = \beta_k \qquad k = 1, 2, \ldots, N \\ A_{kl} = \alpha_{kl}[z_w(1/w)]^{p_{kl}}|_{w=\bar{a}_k} \qquad k = 1, 2, \ldots, N; \quad l = 1, \ldots, P_k, \end{cases} \tag{129}$$

where C, α_{kl} and β_k are constants of integration. It is possible to show that in unstable LG, that is a an exterior problem with growth or an interior problem with shrinking, all the solutions (127) blow up in finite time by forming cusps, generally of the 3/2 kind.

Another interesting class of rational solutions was also found by Kufarev [66] in case there are several sources instead of one, located at z_k with rates Q_k, and $k = 1, 2, \ldots, N$. In this case, the velocity potential (scaled pressure) diverges near z_k logarithmically with coefficients Q_k,

$$-p = Q_k \log(z - z_k) + \text{regular terms} \qquad (\text{when } z \to z_k). \tag{130}$$

In this case, the Laplacian growth equation has a form

$$\mathrm{Im}(\bar{z}_t z_\phi) = \mathrm{Re} \sum_{k=1}^{M} \frac{Q_k}{1 - b_k(t)\,\mathrm{e}^{\mathrm{i}\phi}}, \tag{131}$$

where $b_k(t)$ are time dependent inverse conformal pre-images of sources locations, z_k, so that

$$z_k = z\left(\bar{b}_k^{-1}\right) \qquad k = 1, \ldots, M. \tag{132}$$

In this case, the most general rational solution has a form

$$z = rw + \sum_{k=1}^{N} \sum_{l=1}^{P_k} \frac{A_{kl}}{(w - a_k)^{p_{kl}}} + \sum_{k=1}^{M} \frac{B_k}{w - b_k}. \tag{133}$$

The result of integration is then given by (129), where summations incorporate the last sum in (133), equation (132), and

$$B_k = C_k t [z_w(1/w)]|_{w=\bar{b}_k} \qquad k = 1, 2, \ldots, M, \tag{134}$$

where C_k are additional constants of motion. It is worth to mention that even if the initial configuration $z(0, \mathrm{e}^{\mathrm{i}\phi})$ does not include poles at b_k (knows nothing about sources Q_k at z_k), the solution earns terms with simple poles at b_k immediately from the start, as one can see from the last equation.

Thus, the singularities of any solution can be split into those imposed by the source location (b_k in our case) and those determined by initial configuration, that are a_k.

One should also beware that the interface can reach sources during evolution, thus breaking analyticity by forming a cusp, and after this moment a solution ceases to exist.

Logarithms. In the paper [88] Kufarev and Vinogradov have found a logarithmic class of solutions of (103), which was later rediscovered and studied in detail by several authors [89–93]. This class appeared to be particularly fruitful from both mathematical and physical points of view: besides providing a significant extension from rational solutions, the logarithmic ones are often free of finite-time singularities for an unstable exterior problem, which is the most important for physics. The existence of these solutions for all times allows us to study the long-time asymptotics, which is perhaps the major goal of this research. These so-called multi-logarithmic solutions have a form

$$z = rw + \sum_{k=1}^{N} \alpha_k \log\left(\frac{w}{a_k} - 1\right), \tag{135}$$

where $r\alpha_k$, a_k are parameters (some of them are time dependent), and $|a_k| < 1$ for a conformal mapping from the exterior of the unit disk. Using the method outlined above for rational solutions one could easily figure out that (135) satisfies the LGE (103) with all α_k to be constants in time and the following time dependence of r and a_k:

$$\begin{cases} r^2 + \sum_{k=1}^{N} \sum_{l=1}^{N} \alpha_k \bar{\alpha}_l \log(1 - a_k \bar{a}_l) = C + Qt \\ r/\bar{a}_k + \sum_{l=1}^{N} \alpha_l \log\left(\frac{1}{a_l \bar{a}_k} - 1\right) = \beta_k; \qquad k = 1, 2, \ldots, N, \end{cases} \tag{136}$$

where C and β_k are constants of motion. It is less trivial to show that the solutions (135) may be free of finite-time singularities, but the following example illustrates it well: let us impose a $\mathbb{Z}_N$ symmetry over the system (135) by setting $\alpha_k = \alpha \exp(2\pi \mathrm{i}k/N)$ and $a_k = a \exp(2\pi \mathrm{i}k/N)$, with positive a and α. Then (136) looks significantly simpler,

$$\begin{cases} r^2 + N\alpha^2 \sum_{k=1}^{N} \gamma_k \log(1 - a^2 \gamma_k) = C + Qt \\ r/a + \alpha \sum_{k=1}^{N} \gamma_k \log\left(\frac{1}{a^2 \gamma_k} - 1\right) = \beta, \end{cases} \tag{137}$$

where $\gamma_k = \mathrm{e}^{2\pi \mathrm{i}k/N}$. Equating the derivative of (135) to zero, we find critical points, b_k. As expected, $b_k = b\gamma_k$ and

$$b^N = a^N - \frac{\alpha N a^{N-1}}{r}. \tag{138}$$

Assuming the initial b to be positive, we see that $\dot{b}$ is always positive, if $\dot{a}$ is, which is always the case, since as follows from the second equation in (137),

$$\dot{r} = \left(\frac{r}{a} + 2\alpha N \frac{a^{2N-2}}{1 - a^{2N}}\right)\dot{a}, \tag{139}$$

and therefore $\dot{a}$ is positive, since $\dot{r}$ is. Let us also note that a cannot reach 1 since this would make the RHS of the second equation in (137) infinite, which would contradict to the fact that it is a finite constant. Thus, from

$$0 < b < a < 1, \qquad \dot{a} > 0 \qquad \text{and} \qquad \dot{b} > 0 \tag{140}$$

it follows that critical points and singularities of the conformal map (135) will always stay inside the unit circle, which guarantees the existence of the solution (135) for all times. This simple example illustrates the fact that many of these solutions are free of finite time-singularities (see details in [92, 93]), but the interesting problem of comprehensive classification of initial data for the solutions (135) which do not blow up in finite time still is an open question.

4.4. Mathematical structure of Laplacian growth

4.4.1. Conservation of harmonic moments. The following remarkable property of the Laplacian growth was found by S Richardson in 1972 [94] for a point-like source Q at the origin, for which

$$\nabla^2\phi = \frac{Q}{2\pi}\delta^2(z).$$

He showed that all positive harmonic moments of the viscous domain, $D(t)$,

$$C_k = \int_{D(t)} z^k \,\mathrm{d}x\,\mathrm{d}y, \qquad k = 1, 2, \ldots \tag{141}$$

do not change in time, while the zeroth moment, which is the area of the growing bubble, changes linearly in time,

$$\frac{\mathrm{d}C_k}{\mathrm{d}t} = \oint_{\partial D(t)} z^k V_n \frac{\mathrm{d}l}{\pi} = \oint_{\partial D(t)} (p\partial_n z^k - z^k \partial_n p)\frac{\mathrm{d}l}{\pi}, \tag{142}$$

($\mathrm{d}l$ is an element of arclength) because $V_n = -\partial_n p$ and $p = 0$ along the boundary $\partial D(t)$. By virtue of Gauss' theorem, it equals

$$\int_{D(t)} \nabla(p\nabla z^k - z^k \nabla p)\frac{\mathrm{d}x\,\mathrm{d}y}{\pi} = Q\delta_{k,0}. \tag{143}$$

This property may be used as the definition of the idealized Laplacian growth problem, namely to find an evolution of the domain whose area increases in time, while all positive harmonic moments do not change.

4.4.2. LG and the inverse potential problem. One can easily note that the harmonic moments are the coefficients of the (negative) power expansion of the so-called Cauchy transform $\mathcal{C}_D(z)$ of the domain D, namely

$$\mathcal{C}_D(z) = \frac{1}{\pi}\int_D \frac{\mathrm{d}x'\,\mathrm{d}y'}{z-z'} = \sum_{k=0}^{\infty}\frac{C_k}{z^{k+1}}. \tag{144}$$

Since the Cauchy transform, $\mathcal{C}_D(z)$, is the derivative of the Newtonian potential $\Phi(z)$ created by matter occupied the domain D with a unit density,

$$\Phi(z) = \int_D \log|z-z'|\frac{\mathrm{d}x'\,\mathrm{d}y'}{\pi}, \tag{145}$$

we see a deep connection between the Laplacian growth with the so-called inverse potential problem, asking to find a domain D occupied uniformly by matter which produces a given far-field Newtonian potential. The harmonic moments in this context are multipole moments of this potential. If the domain $D(t)$ grows in accordance with the idealized Laplacian growth, then the potential $\Phi(z)$ changes linearly in time, so (up to a constant),

$$\Phi(t,z) = \frac{Qt}{2\pi}\log|z|, \tag{146}$$

which is a potential of a point-like (increasing in time) mass at the origin.

4.4.3. Laplacian growth in terms of the Schwarz function. Let $F(x,y) = 0$ define an analytic closed Jordanian contour Γ on the plane. Replacing the Cartesian coordinates, x and y, by complex ones, $z = x + \mathrm{i}y$ and $\bar{z} = x - \mathrm{i}y$, one obtains a description of Γ as

$$F\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2\mathrm{i}}\right) = G(z,\bar{z}) = 0. \tag{147}$$

Solving the last equation with respect to $\bar{z}$ one obtains

$$\bar{z} = S(z), \tag{148}$$

when $z \in \Gamma$. The function $S(z)$ is called the Schwarz function of the curve Γ [95]. It is the same mathematical object we encountered in the previous section. This function plays an outstanding role in the theory of quadrature domains (see the following section). It has the following Laurent expansion, valid at least in a strip around the curve Γ:

$$S(z) = \sum_{k=0}^{\infty}\frac{C_k}{z^{k+1}} + \sum_{k=0}^{\infty}kt_k z^{k-1}, \tag{149}$$

where t_k are the external harmonic moments defined as

$$t_k = \frac{1}{\pi k}\int_{D_-}\frac{\mathrm{d}x\,\mathrm{d}y}{z^k}, \qquad k = 1, 2, \ldots, \tag{150}$$

where D_- is the domain complimentary to the domain D.

From (144) and (149) we obtain the connection between the Cauchy transform of a domain with the Schwarz function of its boundary,

$$\mathcal{C}_D(z) = \oint_{\partial D}\frac{S(z')}{z-z'}\frac{\mathrm{d}z'}{2\pi\mathrm{i}}. \tag{151}$$

Rewriting the Laplacian growth dynamics in terms of the Schwarz function, $S(z)$ [87] one obtains

$$\partial_t S(z) = 2\partial_z W, \tag{152}$$

where $W = -p + \mathrm{i}\phi = \log w$ is the complex potential defined earlier. This last form of the Laplacian growth is very instructive. In particular, it helps to understand the origin of constants of integration in all exact solutions of the Laplacian growth equation presented above as a result of direct integrating efforts. Indeed, the RHS in the last equation is analytic in the viscous domain $D(t)$ except a simple pole at the origin (we consider an internal LG problem with a source at the origin). In order for the LHS to satisfy this condition, all the singularities of $S(t, z)$ outside the interface should be constants of motion. At zero the Schwarz function should have a simple pole with a residue (which is the area of the domain $D(t)$) linearly changing in time. This observation can be easily seen as an alternative proof of the Richardson theorem, stated above.

4.4.4. The correspondence of singularities. The Schwarz function is connected to a conformal map $z = f(w)$ from the unit circle to the domain D through the following formula [95]:

$$S(z) = \bar{f}\left(\frac{1}{f^{-1}(z)}\right), \tag{153}$$

where $f^{-1}(z)$ is the inverse of the conformal map $w = f^{-1}(z)$. This formula helps to derive a one-to-one correspondence between singularities of $S(z)$ inside D and $f(w)$ inside the unit circle: if near a singular point a the conformal map $f(w)$ diverges as

$$f(w) = \frac{A}{(w-a)^p}, \tag{154}$$

(here by convention $p = 0$ stands for a logarithmic divergence), then the Schwarz function $S(z)$ diverges near a point $b = f(1/\bar{a})$ with the same power, p, as

$$S(z) = \frac{B}{(z-b)^p}, \tag{155}$$

where

$$A = \left[\frac{\bar{B}}{(-a^2\bar{f}')^p}\right]_{w=1/\bar{a}}. \tag{156}$$

B and b are constants of motions as showed above, thus the last formula together with the relation $b = f(1/\bar{a})$ and the area linearly changing in time and expressed in terms of the parameters of $f(w)$ constitute the whole time dynamics of singularities of $f(t, w)$ [94, 96]. The reader can see the equivalence of these formulae with constants of integration obtained earlier when various classes of exact solutions were derived by direct integration.

4.4.5. A first classification of singularities. As mentioned in the previous sections, the existence of the singular limit was established at the same time with the model [64, 65]. It became a fertile field of study in itself, and led to further developments of the problem [51, 97]. In a series of papers [98–103], the possible boundary singularities were studied, as well as the problem of continuing the solutions for certain classes. It was found that, in the free-space set-up, the generic critical boundary features a cusp at (x_0, y_0), with local geometry of the type

$$(x - x_0)^q \sim (y - y_0)^p, \qquad (p, q) \text{ mutual primes.} \tag{157}$$

The most common cusp is characterized by $q = 2, p = 3$, but $q = 2, p = 5$ can also be obtained fairly easy by choosing proper initial conditions. Very special situations, where a finite-angle geometry is assumed as initial condition were also considered [104].

It was shown be several methods that dynamics can be continued through a cusp of type $(2, 4k + 1), k > 0$ [101, 105].

4.4.6. Hydrodynamics of LG and the singularities of Schwarz function. As indicated above, the Schwarz function encodes information about the conserved moments $\{t_k\}$, through its expansion at infinity [50],

$$S(z) = \frac{t_0}{z} + \sum_{k>0} t_k z^{k-1} + O(z^{-2}). \tag{158}$$

This function is useful when computing averages of integrable analytic functions $f(z)$ over the domain D_+ (an interior domain),

$$\frac{1}{\pi}\int_{D_+} f(z)\,\mathrm{d}x\,\mathrm{d}y = \sum_{k=1}^{N}\sum_{i=1}^{n_k} c_{ik} f^{(i)}(z_k) + \sum_{m=1}^{M}\int_{\gamma_m} h_m(z) f(z)\,\mathrm{d}z, \tag{159}$$

if the function S has poles of order n_k at $z = z_k$ and branch cuts γ_m with jump functions $h_m(z)$. Applying formula (159) for the characteristic function of the domain $f(z) = \chi_{D_+}(z)$ and taking a derivative with respect to t_0, we obtain the relation

$$\frac{\mathrm{d}}{\mathrm{d}t_0}\left[\sum_k \mathrm{Res}\, S(z_k) + \sum_m \int h_m(z)\,\mathrm{d}z\right] = 1, \tag{160}$$

which shows that the singularity data of the Schwarz function in D_+ can be interpreted as giving the location and strength of fluid sources (isolated or line-distributed) [94]. Identifying the 2D uniform measure with another, singular (point or line-distributed) distribution is referred to as *sweeping* of a measure. We will repeatedly encounter this process in the following section. In the case when the Schwarz function is meromorphic in D_+ (it has only isolated poles as singular points), (159) becomes

$$\frac{1}{\pi}\int_{D_+} f(z)\,\mathrm{d}x\,\mathrm{d}y = \sum_{k=1}^{N}\sum_{i=1}^{n_k} c_{ik} f^{(i)}(z_k), \tag{161}$$

and the domain is called a *quadrature domain* [106–110]. Generically, the Schwarz function may have branch cuts in D_+, in which case D_+ is called a *generalized quadrature domain* [111]. This is the typical scenario for our problem. The rigorous theory of quadrature domains is outlined in the following section.

The hydrodynamic interpretation of the Schwarz function arises from (152), which is worthwhile to rewrite here

$$\partial_t S(z) = \partial_z W, \tag{162}$$

after rescaling by 2. Let C be some closed contour, boundary of a domain B, and integrate equation (162) over it. We obtain

$$\partial_t \oint_C S(z)\,\mathrm{d}z = \iint_B \omega\,\mathrm{d}x\,\mathrm{d}y - \mathrm{i}\iint_B \vec{\nabla}\vec{v}\,\mathrm{d}x\,\mathrm{d}y, \tag{163}$$

where $\omega = \partial_y v_x - \partial_x v_y$ is the vorticity field and $\vec{\nabla}\vec{v} = \partial_x v_x + \partial_y v_y$ is the divergence of velocity field. The real part of this identity shows if the flow has zero vorticity, we have

$$\mathrm{Re}\,\partial_t \oint S(z)\,\mathrm{d}z = 0. \tag{164}$$

The imaginary part of (163) illustrates again the interpretation of singularity set of $S(z)$ as sources of *water* (which occupies D_+ in a canonical Laplacian growth formulation, while the exterior domain, D_-, is occupied by a viscous fluid, which we call *oil* [50]): assume that the contour C in (163) encircles the droplet without crossing any other branch cuts, then the contour integral may be performed using Cauchy's theorem, giving the total flux of water,

$$\iint_B \vec{\nabla} \cdot \vec{v}\, \mathrm{d}x\, \mathrm{d}y = Q = 1. \tag{165}$$

We note here that equation (162) implies existence of a closed form

$$\mathrm{d}\Omega = S\, \mathrm{d}z + W\, \mathrm{d}t, \tag{166}$$

whose primitive Ω has for real part the *Baiocchi transform* of p,

$$\mathrm{Re}\, \Omega = -\int_0^t p(z, \tau)\, \mathrm{d}\tau. \tag{167}$$

One can see that $\mathrm{Re}\, \Omega$ coincides with the potential Φ introduced earlier. From the continuity equation for water $\dot{\rho} + \vec{\nabla}\vec{v} = 0$ and the Darcy law for water (opposite to oil) $\vec{v} = \nabla p$, we obtain for the time evolution of water density at a given point z,

$$\dot{\rho} = -\Delta p \Rightarrow \rho(z, t) = \rho(z, 0) - \Delta \int_0^t p(z, \tau)\, \mathrm{d}\tau. \tag{168}$$

Equation (168) may be immediately generalized in a weak sense, replacing the water density by the characteristic function of the domain D_+, $\rho \to \chi_{D_+}$, which shows that the Baiocchi transform $\mathrm{Re}\, \Omega$ may be interpreted as the electrostatic potential giving the growth of the water domain.

Similarly, applying an antiholomorphic derivative to (162), we obtain

$$\vec{\nabla} \cdot \vec{v} + \mathrm{i}\omega = -\Delta p + \mathrm{i}\Delta\phi, \tag{169}$$

so that the imaginary part of the form Ω can be considered an electrostatic potential for the time integral of vorticity at a given point z,

$$\Delta\, \mathrm{Im}\, \Omega(z, t) = \int_0^t \omega(z, \tau)\, \mathrm{d}\tau. \tag{170}$$

4.4.7. Variational formulation of Hele–Shaw dynamics. Formula (159) has another physical interpretation, which we explore in this section. Besides hydrodynamics, it also allows us to describe the droplet through a variational (minimization) formulation, which will become very relevant when considering the singular limit.

Consider the case when the Schwarz function has only simple poles $\{z_k\}$ and cuts at $\{\gamma_m\}$, with residues $\mathrm{Res}\ S(z_k)$ and jump functions $h_m(z)$, inside the droplet. A simple calculation shows that these singular points constitute electrostatic sources for the potential $\mathrm{Re}\, \Omega$,

$$\Delta\, \mathrm{Re}\, \Omega(z) = \sum_k \mathrm{Res}\, S(z_k)\delta(z - z_k) + \sum_m \int_{\gamma_m} h_m(\zeta)\delta(z - \zeta)\, \mathrm{d}\zeta. \tag{171}$$

If we apply (159) to all positive powers z^k, $k \geqslant 0$, we conclude that the singular distribution $\{z_k\}, \{\gamma_m\}$ and the uniform distribution $\rho(z) = \chi_{D_+}(z)$ have the same interior harmonic moments $v_k = \langle z^k \rangle$, $k \geqslant 0$. Thus, they create the same electrostatic potential outside the droplet. It is therefore possible to substitute the actual singular distribution $\{z_k\}, \{\gamma_m\}$ with the smooth, uniform distribution $\rho(z)$ in calculations related to the exterior potential. Beyond the mathematical equivalence, however, this fact has an important physical interpretation, whose full meaning will become apparent in the critical limit: when one more quantum of water is pumped into the droplet, it first appears as a new singular point of the Schwarz function (a δ-function singularity). After a certain time, though, the droplet adjusts to the new area (subject to the constraints given by the fixed exterior harmonic moments), and reaches its new shape (with uniform density of water inside). Therefore, we can say that the singular distribution $\{z_k\}$, $\{\gamma_m\}$ represents the fast-time distribution of sources of water, while the uniform distribution $\rho(z)$ is the long-time, equilibrium distribution of the same amount of water. When the dynamics becomes fully non-equilibrium (after the cusp formation), this equivalence breaks down, and the correct distribution to work with is the set of poles (cuts) of the Schwarz function. In that case, the issue becomes solving the Poisson problem $\Delta \operatorname{Re} \Omega = \sum_k \operatorname{Res} S(z_k)\delta(z - z_k)$, and finding the actual (time-dependent) location of the distribution of charges $z_k(t)$, subject to usual conditions for the electrostatic potential $\Delta \operatorname{Re} \Omega$.

In the equilibrium case, however, it is appropriate to work with the smooth distribution $\rho(z)$. Since the actual electrostatic potential $\Delta \operatorname{Re} \Omega$ contains the regular expansion $V(z) = \sum_k t_k z^k$, we also add it to the contribution due to the distribution $\rho(z)$. We obtain for the total potential,

$$\Phi(z, \bar{z}) = \int_{D_+} \rho(\zeta) \log|z - \zeta|^2 \, \mathrm{d}^2\zeta + V(z) + \overline{V(z)}. \tag{172}$$

Inside the droplet, this potential solves the Poisson problem $\Delta\Phi(z, \bar{z}) = \rho(z) = 1$, and on the boundary it creates the electric field $E(z) = \bar{\partial}\Phi = z$. This means that inside the droplet, this potential is actually equal to $|z|^2$. Therefore, the problem of finding the actual shape of a droplet of area t_0 and harmonic moments $\{t_k\}$ can be stated as

Find the domain D_+ of area t_0 such that $\Phi(z, \bar{z}) = |z|^2$ on D_+.

Since $\rho(z)$ is the characteristic function of D_+, we may also write this problem in the variational form

$$\frac{\delta}{\delta\rho(z)} \int_{D_+} \rho(z) \left[|z|^2 - V(z) - \overline{V(z)} - \int_{D_+} \rho(\zeta) \log|z - \zeta|^2 \, \mathrm{d}^2\zeta \right] \mathrm{d}^2 z = 0.$$

This equation is simply the minimization condition for the total energy of a distribution of charges $\rho(z)$, in the external potential $W(z, \bar{z}) = -|z|^2 + V(z) + \overline{V(z)}$. Therefore, the equilibrium (long-time limit) distribution of water has the usual interpretation of minimizing the total electrostatic energy of the system. However, when the system is not in equilibrium, this criterion cannot be used to select the solution.

5. Quadrature domains

We have seen in the previous sections that polynomial or rational conformal mappings from the disk have as images planar domains which are relevant for the Laplacian growth (with finitely many sources). The domains in question were previously and independently studied by mathematicians, for at least two separate motivations. First they have appeared in the work of Aharonov and Shapiro, on extremal problems of univalent function theory [112]. About the same time, these domains have been isolated by Makoto Sakai in his potential theoretic work [113]. These domains, known today as *quadrature domains*, carry Gaussian-type quadrature formulae which are valid for several classes of functions, like integrable analytic, harmonic and sub-harmonic functions. The geometric structure of their boundary, the qualitative properties of their boundary defining function and dynamics under the Laplacian growth law are well understood. The reader can consult the recent collection of papers [114] and the survey [115]. The present section contains a general view of the theory of quadrature domains, with special emphasis of a matrix model realization of their defining function.

This section is organized in the following way: after presenting the theory of quadrature domains for subharmonic and analytic functions, we give an overview of the (inverse) Markov problem of moments, followed by its analogue in two dimensions, which is based on the notion of exponential transform in the complex plane. The following sections illustrate the reconstruction algorithm for the shape of a droplet, and point to a few essential properties of the problem for signed measures.

5.1. Quadrature domains for subharmonic functions

Let φ be a subharmonic function defined on an open subset of the complex plane, that is $\Delta\varphi \geqslant 0$, in the sense of distributions, or the submeanvalue property

$$\varphi(a) \leqslant \frac{1}{|B(a,r)|}\int_{B(a,r)} \varphi \, \mathrm{dA}$$

holds for any disc centered at a, of radius r, $B(a,r)$ contained in the domain of definition of φ. Henceforth $\mathrm{d}A$ denotes Lebesgue planar measure. Thus, with $\Omega = B(a,r)$, $c = |B(a,r)| = \pi r^2$ and $\mu = c\delta_a$ there holds

$$\int \varphi \, \mathrm{d}\mu \leqslant \int_{\Omega} \varphi \, \mathrm{dA} \tag{173}$$

for all subharmonic functions φ in Ω. This set of inequalities is encoded in the definition that Ω is a *quadrature domain for subharmonic functions* with respect to μ [113], and it expresses that $\Omega = B(a,r)$ is a *swept out* version of the measure $\mu = c\delta_a$. If c increases the corresponding expansion of Ω is a simple example of Hele–Shaw evolution, or Laplacian growth, as we have seen in the previous section.

The above can be repeated with finitely many points, i.e., with μ of the form

$$\mu = c_1\delta_{a_1} + \cdots + c_n\delta_{a_n}, \tag{174}$$

$a_j \in \mathbb{C}, c_j > 0$: there always exists a unique (up to nullsets) open set $\Omega \subset \mathbb{C}$ such that (173) holds for all φ subharmonic and integrable in Ω. One can think of it as the union $\bigcup_{j=1}^{n} B(a_j, r_j), r_j = \sqrt{c_j/\pi}$, with all multiple coverings smashed out to a singly covered set, Ω. In particular, $\bigcup_{j=1}^{n} B(a_j, r_j) \subset \Omega$.

The above sweeping process, $\mu \mapsto \Omega$, or better $\mu \mapsto \chi_\Omega \cdot (\mathrm{dA})$, called *partial balayage* [113, 116, 117], applies to quite general measures $\mu \geqslant 0$ and can be defined in terms of a natural energy minimization: given μ, $\nu = \chi_\Omega \cdot (\mathrm{dA})$ will be the unique solution of

$$\mathrm{Minimize}_\nu \|\mu - \nu\|_e^2 \quad \text{s.t.} \quad \nu \leqslant \mathrm{dA}, \qquad \int \mathrm{d}\nu = \int \mathrm{d}\mu.$$

Here $\|\cdot\|_e$ is the energy norm,

$$\|\mu\|_e^2 = (\mu,\mu)_e, \qquad \text{with} \qquad (\mu,\nu)_e = \frac{1}{2\pi}\int \log\frac{1}{|z-\zeta|}\,\mathrm{d}\mu(z)\,\mathrm{d}\nu(\zeta).$$

If μ has infinite energy, like in (174), one minimizes $-2(\mu,\nu) + \|\nu\|_e^2$ instead of $\|\mu-\nu\|_e^2$, which can always be given a meaning [26].

By choosing

$$\varphi(\zeta) = \pm\log|z-\zeta|$$

in equation (173), the plus sign allowed for all $z \in \mathbb{C}$, the minus sign allowed only for $z \notin \Omega$, one gets the following statements for potentials:

$$\begin{cases} U^\mu \geqslant U^\Omega & \text{in all } \mathbb{C}, \\ U^\mu = U^\Omega & \text{outside } \Omega. \end{cases} \tag{175}$$

Here

$$U^{\mu}(z) = \frac{1}{2\pi}\int \log\frac{1}{|z-\zeta|}\,\mathrm{d}\mu(\zeta)$$

denotes the logarithmic potential of the measure μ, and $U^{\Omega} = U^{\chi_\Omega \cdot \mathrm{dA}}$. In particular, the measures μ and $\chi_\Omega \cdot$ (dA) are gravi-equivalent outside Ω. By an approximation argument, (175) is actually equivalent to (173).

Let us consider now an integrable harmonic function h, defined in the domain Ω. Since both $\varphi = \pm h$ are subharmonic functions, we find

$$\int_{\Omega} h\,\mathrm{d}A = \int h\,\mathrm{d}\mu = \sum_{j=1}^{n} c_j h(a_j). \qquad (176)$$

That is, a Gaussian-type quadrature formula, with nodes $\{a_j\}$ and weights $\{c_j\}$ holds. We say in this case that Ω is a *quadrature domain for harmonic functions*. Similarly, one defines a quadrature domain for complex analytic functions, and it is worth mentioning that the inclusions {QD for subharmonic functions} $\subset$ {QD for harmonic functions} $\subset$ {QD for analytic functions} are strict, see for details [113].

Recall that for a given positive measure σ on the line, rapidly decreasing at infinity, the zeros of the Nth orthogonal polynomial are the nodes of a Gauss quadrature formula, valid only for polynomials of degree $2N-1$. The difference above is that the same finite quadrature formula is valid, in the plane, for an infinite-dimensional space of functions. A common feature of the two scenarios, which will be clarified in the sequel, is the link between quadrature formulae (on the line or in the plane) and spectral decompositions (of Jacobi matrices, respectively hyponormal operators).

Let $K = \operatorname{conv}\operatorname{supp}\mu$ be the convex hull of the support of μ, i.e., the convex hull of the points $a_1, \ldots, a_n$. As mentioned, Ω can be thought of as smashed out version of $\bigcup_{j=1}^{n} B(a_j, r_j)$. The geometry of Ω which this enforces is expressed in the following sharp result [116, 118, 119]: assume that Ω satisfies (173) for a measure $\mu \geqslant 0$ of the form (174). Then:

(i) $\partial\Omega$ may have singular points (cusps, double points, isolated points), but they are all located inside K. Outside K, $\partial\Omega$ is smooth algebraic.
For $z \in \partial\Omega\backslash K$, let N_z denote the inward normal of $\partial\Omega$ at z (well defined by (i)).
(ii) For each $z \in \partial\Omega\backslash K$, N_z intersects K.
(iii) For $z, w \in \partial\Omega\backslash K$, $z \neq w$, N_z and N_w do not intersect each other before they reach K. Thus $\Omega\backslash K$ is the disjoint union of the inward normals from $\partial\Omega\backslash K$.
(iv) There exist $r(z) > 0$ for $z \in K\cap\Omega$ such that

$$\Omega = \bigcup_{z\in K\cap\Omega} B(z, r(z)).$$

(Statement (iv) is actually a consequence of (iii).)

To better connect our discussion with the moving boundaries encountered in the first part of this survey, we add the following remarks. Since Ω is uniquely determined by (a_j, c_j) one can steer Ω by changing the c_j (or a_j). Such deformations are of Hele–Shaw type, as can be seen by the following computation, which applies in more general situations: Hele–Shaw evolution $\Omega(t)$ corresponding to a point source at $a \in \mathbb{C}$ ('injection of fluid' at a) means that $\Omega(t)$ changes by $\partial\Omega(t)$ moving in the outward normal direction with speed

$$-\frac{\partial G_{\Omega(t)}(\cdot, a)}{\partial n}.$$

Here $G_\Omega(z,a)$ denotes the Green function of the domain Ω. If φ is subharmonic in a neighborhood of $\overline{\Omega(t)}$ then, as a consequence of $G_\Omega(\cdot,a) \geqslant 0, G_\Omega(\cdot,a) = 0$ on $\partial\Omega$ and $-\Delta G_\Omega(\cdot,a) = \delta_a$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} \varphi\,\mathrm{dA} = \int_{\partial\Omega(t)} (\text{speed of } \partial\Omega(t) \text{ in normal direction})\varphi\,\mathrm{d}s$$

$$= -\int_{\partial\Omega(t)} \frac{\partial G_{\Omega(t)}(\cdot,a)}{\partial n}\varphi\,\mathrm{d}s = -\int_{\partial\Omega(t)} \frac{\partial\varphi}{\partial n} G_{\Omega(t)}(\cdot,a)\,\mathrm{d}s$$

$$-\int_{\Omega(t)} \varphi\Delta G_{\Omega(t)}(\cdot,a)\,\mathrm{dA} + \int_{\Omega(t)} G_{\Omega(t)}(\cdot,a)\Delta\varphi\,\mathrm{dA} \geqslant \varphi(a).$$

Hence, integrating from $t = 0$ to an arbitrary $t > 0$,

$$\int_{\Omega(t)} \varphi\,\mathrm{dA} \geqslant \int_{\Omega(0)} \varphi\,\mathrm{dA} + t\varphi(a),$$

telling that if $\Omega(0)$ is a quadrature domain for μ then $\Omega(t)$ is a quadrature domain for $\mu + t\delta_a$.

We remark that quadrature domains for subharmonic functions can be defined in any number of variables, but then much less of their qualitative properties are known, see for instance [114].

5.2. *Quadrature domains for analytic functions*

Critical for our study is the regularity and algebraicity of the boundary of quadrature domains for analytic functions. This was conjectured in the early works of Aharonov and Shapiro, and proved in full generality by Gustafsson [108]. A description of the possible singular points in the boundary of a quadrature domain was completed by Sakai [98–100].

Assume that the quadrature domain for analytic functions Ω has a sufficiently smooth boundary Γ. Let us consider the Cauchy transform of the area mass, uniformly distributed on Ω,

$$C(z) = \frac{-1}{\pi}\int_\Omega \frac{\mathrm{d}A(w)}{w - z}.$$

This is an analytic function on the complement of $\overline{\Omega}$, which is continuous (due to the Lebesgue integrability of the kernel) on the whole complex plane. In addition, the quadrature identity implies

$$C(z) = \sum_{j=1}^{n} \frac{c_j}{\pi(z - a_j)}, \qquad z \in \mathbb{C}\setminus\overline{\Omega}.$$

From the Stokes formula,

$$C(z) = \frac{-1}{2\pi\mathrm{i}}\int_\Gamma \frac{\overline{w}\mathrm{d}w}{w - z}, \qquad z \in \mathbb{C}\setminus\overline{\Omega}.$$

Therefore, by standard arguments in function theory one proves that the continuous function $w \mapsto \overline{w}$ extends meromorphically from Γ to Ω. The poles of this meromorphic extension coincide with the quadrature nodes.

The converse also holds, in virtue of Cauchy's formula: if f is an integrable analytic function in Ω, then

$$\int_\Omega f\,\mathrm{dA} = \int_\Gamma f(w)\overline{w}\mathrm{d}w = \sum_{j=1}^{n} c_j f(a_j).$$

Thus, we recover the following fundamental observation: *if Ω is a bounded planar domain with sufficiently smooth boundary Γ, then Ω is a quadrature domain for analytic functions if and only if the function $w \mapsto \overline{w}$ extends meromorphically from Γ to Ω.*

Note that above, and elsewhere henceforth, we do not assume that the weights in the quadrature formula for analytic functions are positive. In this way we recover the fact (already noted in the previous sections) that quadrature domains for analytic functions are characterized by a meromorphic Schwarz function, usually denoted $S(z)$. A second departure from the quadrature domains for subharmonic functions is that the quadrature data (a_j, c_j) *do not determine* the quadrature domain for analytic functions. Indeed, consider the annulus $A_{r,R} = \{z, r < |z| < R\}$. Then

$$\int_{A_{r,R}} f \, \mathrm{d}A = \pi(R^2 - r^2) f(0),$$

for all analytic, integrable functions f in $A_{r,R}$.

The question how weak the smoothness assumption on the boundary Γ can be to insure the use of the above arguments has a long history by itself, and we do not enter into its details. Simply the existence of the quadrature formula and the fact that the boundary is a mere continuum implies, via quite sophisticated techniques, the regularity of Γ. See for instance [98, 120].

The Schwarz function is a central character in our story. It can also be related to the logarithmic potentials introduced in the previous subsection. More specifically, given any measure μ as in (174) and any open set Ω containing supp μ, define (as distributions in all $\mathbb{C}$)

$$u = U^{\mu} - U^{\Omega}, \qquad S(z) = \overline{z} - 4\frac{\partial u}{\partial z}.$$

Then

$$\Delta u = \chi_{\Omega} - \mu, \qquad \frac{\partial S}{\partial \overline{z}} = 1 - \chi_{\Omega} + \mu.$$

Note that with μ of the form (174) w is harmonic in Ω except for poles at the points a_j and that in particular, $S(z)$ is meromorphic in Ω.

It is clear from (175) that Ω is a subharmonic quadrature domain for μ if and only if $u \geqslant 0$ everywhere and $u = 0$ outside Ω. Then also $\nabla u = 0$ outside Ω. Similarly, the criterion for Ω being a quadrature domain for harmonic functions is that merely $u = \nabla u = 0$ on $\mathbb{C}\backslash\Omega$. (The vanishing of the gradient is a consequence of the vanishing of u, except at certain singular points on the boundary.) To be a quadrature domain for analytic functions it is enough that just the gradient vanishes, or better in the complex-valued case, that $\frac{\partial u}{\partial z} = 0$ on $\mathbb{C}\backslash\Omega$ (or just on $\partial\Omega$).

Gustafsson's innovative idea, to use the Schottky double of the domain, can be summarized as follows. Let Ω be a bounded quadrature domain for analytic functions, with boundary Γ. We consider a second copy $\tilde{\Omega}$ of Ω, endowed with the anti-conformal structure, and 'glue' them into a compact Riemann surface

$$X = \Omega \cup \Gamma \cup \tilde{\Omega}.$$

This (connected) Riemann surface carries two meromorphic functions:

$$f(z) = \begin{cases} S(z), & z \in \Omega \\ \overline{z}, & z \in \tilde{\Omega} \end{cases}, \qquad g(z) = \begin{cases} z, & z \in \Omega \\ \overline{S(z)}, & z \in \tilde{\Omega}. \end{cases}$$

Any pair of meromorphic functions on X is algebraically dependent, that is, there exists a polynomial $Q(z, w)$ with the property $Q(g, f) = 0$, and in particular

$$Q(z, S(z)) = Q(z, \overline{z}) = 0, \qquad z \in \Gamma.$$

The involution (flip from one side to its mirror symmetric) on X yields the Hermitian structure of Q,

$$Q(z,w) = \sum_{i,j}^{n} a_{ij} z^i w^j, \qquad a_{ij} = \overline{a_{ji}}.$$

One also proves by elementary means of Riemann surface theory that Q is irreducible, and moreover, its leading part is controlled by the quadrature identity data,

$$Q(z,\bar{z}) - |P(z)|^2 = O(z^{n-1}, \bar{z}^{n-1}),$$

where

$$P(z) = (z-a_1)(z-a_2)\cdots(z-a_n).$$

This Riemann surface is the continuum limit of the spectral curve (67), for $N \to \infty$. Following Gustafsson [108], we note a surprising result:

(a) The boundary of a quadrature domain for analytic functions is a real algebraic, irreducible curve.

(b) In every conformal class of finitely connected planar domains there exists a quadrature domain.

(c) Every bounded planar domain can be approximated in the Haudorff distance by a sequence of quadrature domains.

The last two assertions are proven in Gustafsson's influential thesis [108]. Recently, considerable progress was made in the construction of multiply connected quadrature domains, see [114, 121, 121].

It is important to point out that not every domain bounded by an algebraic curve is an algebraic domain in the above sense. In general, if a domain $\Omega \subset \mathbb{C}$ is bounded by an algebraic curve $Q(z,\bar{z}) = 0$ (Q a polynomial with Hermitian symmetry), then one can associate two compact symmetric Riemann surfaces to it: one is the Schottky double of Ω and the other is the Riemann surface classically associated with the complex curve $Q(z,w) = 0$. For the latter the involution is given by $(z,w) \mapsto (\overline{w}, \bar{z})$. In the case of *algebraic domains* (this is another circulating name for quadrature domains for analytic functions), and only in that case, the two Riemann surfaces canonically coincide: the lifting

$$z \mapsto (z, S(z))$$

from Ω to the locus of $Q(z,w) = 0$ extends to the Schottky double of Ω and then gives an isomorphism, respecting the symmetries, between the two Riemann surfaces.

As a simple example, the Schottky double of the simply connected domain

$$\Omega = \{z = x + \mathrm{i}y \in \mathbb{C} : x^4 + y^4 < 1\}$$

has genus zero, while the Riemann surface associated with the curve $x^4 + y^4 = 1$ has genus 3. Hence they cannot be identified, and in fact Ω is not an algebraic domain.

Other ways of characterizing algebraic domains, by means of rational embeddings into n dimensional projective space, are discussed in [122].

5.3. Markov's moment problem

We pause for a while the main line of our story, to connect the described phenomenology with a classical, beautiful mathematical construct due to A A Markov, all gravitating around moment problems for bounded functions.

The classical *L-problem of moments* (also known as *Markov's moment problem*) offers a good theoretical framework for reconstructing extremal measures μ from their moments, or equivalently, from the germ at infinity of some of their integral transforms. The material below is classical and can be found in the monographs [123, 124]. We present only a simplified version of the abstract L-problem, well adapted to the main themes of this survey.

Let K be a compact subset of $\mathbb{R}^n$ with interior points and let $A \subset \mathbb{N}^n$ be a finite subset of multi-indices. We are interested in the set Σ_A of moment sequences $a(f) = (a_\sigma(f))_{\sigma\in A}$,

$$a_\sigma(f) = \int_K x^\sigma f(x)\,\mathrm{d}x, \qquad \sigma \in A,$$

of all measurable functions $f : K \longrightarrow [0, 1]$. Regarded as a subset of $\mathbb{R}^{|A|}$, Σ_A is a compact convex set. An $L^1 - L^\infty$ duality argument (known as the abstract L-problem of moments) shows that every extremal point of Σ_A is a characteristic function of the form $\chi_{\{p<\gamma\}}$, where we denote

$$\{p < \gamma\} = \{x \in K;\ p(x) < \gamma\}.$$

Above γ is a real constant and p is an A-polynomial with real coefficients, that is $p(x) = \sum_{\sigma\in A} c_\sigma x^\sigma$. Indeed, to find the special form of the extremal functions f, one has to analyze when the inequality

$$\int_K p(x) f(x)\,\mathrm{d}x \leqslant \|p\|_1 \|f\|_\infty = \int_K |p(x)|\,\mathrm{d}x$$

is an equality. For a complete proof the reader can consult Krein and Nudelman's monograph [124].

As a consequence, the above description of the extremal points in the moment set Σ_A implies the following remarkable uniqueness theorem due to Akhiezer and Krein:

For each characteristic function χ of a level set in K of an A-polynomial there exists exactly one class of functions f in $L^\infty(K)$ satisfying $a(f) = a(\chi)$. For a non-extremal point $a(f) \in \Sigma_A$ there are infinitely many non-equivalent classes in $L^\infty(K)$ having the same A-moments.

Let us consider a simple example,

$$K = \{(x, y);\ x^2 + y^2 \leqslant 1\} \subset \mathbb{R}^2,$$

and

$$\Omega_+ = \{(x, y) \in K;\ x > 0, y > 0\}, \qquad \Omega_- = \{(x, y) \in K;\ x < 0, y < 0\}.$$

The reader can prove by elementary means that the sets $\Omega_\pm$ cannot be defined in the unit ball K by a single polynomial inequality. On the other hand, the set

$$\Omega = \Omega_+ \cup \Omega_- = \{(x, y);\ xy > 0\}$$

is defined by a single equation of degree 2.

Thus, no matter how the finite set of indices $A \subset \mathbb{N}^2$ is chosen, there is a continuum $f_s, s \in \mathbb{R}$, of essentially distinct measurable functions $f_s : K \longrightarrow [0, 1]$ possessing the same A-moments,

$$\int_K x^{\sigma_1} y^{\sigma_2} f_s(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\Omega_+} x^{\sigma_1} y^{\sigma_2}\,\mathrm{d}x\,\mathrm{d}y, \qquad s \in \mathbb{R}, \sigma \in A.$$

In contrast, if the set of indices A contains $(0, 0)$ and $(1, 1)$, then for every measurable function $f : K \longrightarrow [0, 1]$ satisfying

$$\int_K x^{\sigma_1} y^{\sigma_2} f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega x^{\sigma_1} y^{\sigma_2}\,\mathrm{d}x\,\mathrm{d}y, \qquad \sigma \in A,$$

we infer by Akhiezer and Krein's theorem that $f = \chi_\Omega$, almost everywhere.

On a more theoretical side, we can interpret Akhiezer and Krein's theorem in terms of geometric tomography, see [125]. Fix a unit vector $\omega \in \mathbb{R}^n$, $\|\omega\| = 1$, and let us consider the parallel Radon transform of a function $f : K \longrightarrow [0, 1]$, along the direction ω,

$$(Rf)(\omega, s) = \int_{\langle x,\omega\rangle = s} f(x)\,\mathrm{d}x.$$

Accordingly, the kth moment in the variable s of the Radon transform is, for a sufficiently large constant M,

$$\int_{-M}^{M} (Rf)(\omega, s)s^k\,\mathrm{d}s = \int_K \langle x, \omega\rangle^k f(x)\,\mathrm{d}x \tag{177}$$

$$= \sum_{|\sigma|=k} \frac{|\sigma|!}{\sigma!} \int_K x^\sigma \omega^\sigma f(x)\,\mathrm{d}x = \sum_{|\sigma|=k} \frac{|\sigma|!}{\sigma!} \omega^\sigma a_\sigma(f). \tag{178}$$

Since there are $N(n, d) = C_{n+d}^n$ linearly independent polynomials in n variables of degree less than or equal to d, a Vandermonde determinant argument shows, via the above formula, that the same number of different parallel projections of the 'shade' function $f : K \longrightarrow [0, 1]$, determine, via a matrix inversion, all moments,

$$a_\sigma(f), \qquad |\sigma| \leqslant d.$$

The converse also holds, by formula (178). These transformations are known and currently used in image processing, see for instance [126] and the references cited there.

In conclusion, Akhiezer and Krein's theorem asserts then that in the measurement process

$$f \mapsto ((Rf)(\omega_j, s))_{j=1}^{N(n,d)} \mapsto (a_\sigma(f))_{|\sigma|\leqslant d}$$

only black and white pictures, delimited by a single algebraic equation of degree less than or equal to d, can be exactly reconstructed. Even when these uniqueness conditions are met, the details of the reconstruction from moments are delicate. We shall see some examples in the following sections.

5.3.1. Markov's extremal problem and the phase shift. By going back to the source and dropping a few levels of generality, we recall Markov's original moment problem and some of its modern interpretations. Highly relevant for our 'quatization' approach to moving boundaries of planar domains is the matrix interpretation we will describe for Markov's moment. Again, this material is well exposed in the monograph by Krein and Nudelman [124].

Let us consider, for a fixed positive integer n, the L-moment problem on the line,

$$a_k = a_k(f) = \int_{\mathbb{R}} t^k f(t)\,\mathrm{d}t, \qquad 0 \leqslant k \leqslant 2n,$$

where the unknown function f is measurable, admits all moments up to degree $2n$ and satisfies

$$0 \leqslant f \leqslant L, \text{ a.e.}$$

As noted by Markov, the next formal series transform is quite useful for solving this question:

$$\exp\left[\frac{1}{L}\left(\frac{a_0}{z} + \frac{a_1}{z^2} + \cdots \frac{a_{2n}}{z^{2n+1}}\right)\right] = 1 + \frac{b_0}{z} + \frac{b_1}{z^2} + \cdots. \tag{179}$$

Remark that, although the series under the exponential is finite, the resulting one might be infinite.

The following result is classical, see for instance [123, pp 77–82]. Its present form was refined by Akhiezer and Krein; partial similar attempts are due, among others, to Boas, Ghizzetti, Hausdorff, Kantorovich, Verblunsky and Widder, see [123, 124].

(Markov) Let $a_0, a_1, \ldots, a_{2n}$ be a sequence of real numbers and let $b_0, b_1, \ldots$ be its exponential L-transform. Then there is an integrable function f, $0 \leqslant f \leqslant L$, possessing the moments $a_k(f) = a_k$, $0 \leqslant k \leqslant 2n$, if and only if the Hankel matrix $(b_{k+l})_{k,l=0}^n$ is non-negative definite. Moreover, the solution f is unique if and only if $\det(b_{k+l})_{k,l=0}^n = 0$. In this case the function f/L is the characteristic function of a union of at most n bounded intervals.

The reader will recognize above a concrete validation of the abstract moment problem discussed in the previous section.

In order to better understand the nature of the L-problem, we interpret below the exponential transform from two different and complementary points of view. For simplicity we take the constant L to be equal to 1 and consider only compactly supported originals f, due to the fact that the extremal solutions have anyway compact support. Let μ be a positive Borel measure on $\mathbb{R}$, with compact support. Its Cauchy transform

$$F(z) = 1 - \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{t - z}$$

provides an analytic function on $\mathbb{C}\backslash\mathbb{R}$ which is also regular at infinity, and has the normalizing value 1 there. The power expansion, for large values of $|z|$, yields the generating moment series of the measure μ,

$$F(z) = 1 + \frac{b_0(\mu)}{z} + \frac{b_1(\mu)}{z^2} + \frac{b_2(\mu)}{z^3} + \cdots.$$

On the other hand,

$$\operatorname{Im} F(z) = -\operatorname{Im} z \int \frac{\mathrm{d}\mu(t)}{|t - z|^2},$$

whence

$$\operatorname{Im} F(z) \operatorname{Im} z < 0, \qquad z \in \mathbb{C}\backslash\mathbb{R}.$$

Thus the main branch of the logarithm $\log F(z)$ exists in the upper half-plane and its imaginary part, equal to the argument of $F(z)$, is bounded from below by $-\pi$ and from above by 0. According to Fatou's theorem, the non-tangential boundary limits

$$f(t) = \lim_{\epsilon \to 0} \frac{-1}{\pi} \operatorname{Im} \log F(t + \mathrm{i}\epsilon)$$

exist and produce a measurable function with values in the interval [0, 1]. According to Riesz–Herglotz formula for the upper-half plane, we obtain

$$\log F(z) = -\int_{\mathbb{R}} \frac{f(t)\,\mathrm{d}t}{t - z}, \qquad z \in \mathbb{C}\backslash\mathbb{R}.$$

Or equivalently,

$$F(z) = \exp\left[-\int_{\mathbb{R}} \frac{f(t)\,\mathrm{d}t}{t - z}\right].$$

One step further, let us consider the Lebesgue space $L^2(\mu)$ and the bounded self-adjoint operator $A = M_t$ of multiplication by the real variable. The vector $\xi = \mathbf{1}$ corresponding to the constant function 1 is A-cyclic, and according to the spectral theorem

$$\int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{t - z} = \langle (A - z)^{-1}\xi, \xi\rangle, \qquad z \in \mathbb{C}\backslash\mathbb{R}.$$

As a matter of fact an arbitrary function F which is analytic on the Riemann sphere minus a compact real segment, and which maps the upper/lower half-plane into the opposite half-plane has one of the above forms. These functions are known in rational approximation theory as *Markov functions*.

In short, putting together the above comments we can state the following result: the canonical representations,

$$F(z) = 1 - \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{t-z} = \exp\left(-\int_{\mathbb{R}} \frac{f(t)\,\mathrm{d}t}{t-z}\right) = 1 - \langle (A-z)^{-1}\xi, \xi\rangle,$$

establish constructive equivalences between the following classes:

(a) Markov's functions $F(z)$;
(b) Positive Borel measures μ of compact support on $\mathbb{R}$;
(c) Functions $f \in L^{\infty}_{\mathrm{comp}}(\mathbf{R})$ of compact support, $0 \leqslant f \leqslant 1$;
(d) Pairs (A, ξ) of bounded self-adjoint operators with a cyclic vector ξ.

The extremal solutions correspond, in each case exactly, to:

(a) Rational Markov functions F;
(b) Finitely many point masses μ;
(c) Characteristic functions f of finitely many intervals;
(d) Pairs (A, ξ) acting on a finite-dimensional Hilbert space.

For a complete proof see for instance chapter VIII of [127] and the references cited there. The above dictionary is remarkable in many ways. Each of its terms has intrinsic values. They were long ago recognized in moment problems, rational approximation theory or perturbation theory of self-adjoint operators.

For instance, when studying the change of the spectrum under a rank-1 perturbation $A \mapsto B = A - \xi\langle\cdot, \xi\rangle$ one encounters the *perturbation determinant*:

$$\Delta_{A,B}(z) = \det[(A - \xi\langle\cdot,\xi\rangle - z)(A-z)^{-1}] = 1 - \langle (A-z)^{-1}\xi, \xi\rangle.$$

The above exponential representation leads to the *phase-shift* function $f_{A,B}(t) = f(t)$,

$$\Delta_{A,B}(z) = \exp\left[-\int_{\mathbb{R}} \frac{f_{A,B}(t)\,\mathrm{d}t}{t-z}\right].$$

The phase shift of, in general, a trace-class perturbation of a self-adjoint operator has certain invariance properties; it reflects by fine qualitative properties the nature of change in the spectrum. The theory of perturbation determinants and of the phase shift is nowadays well developed, mainly for its applications to quantum physics, see [128, 129].

The reader will recognize above an analytic continuation in the complex plane of the real exponential transform

$$F(x) = E_f(x) = \exp\left[-\int_{\mathbb{R}} \frac{f(t)\,\mathrm{d}t}{|t-x|}\right],$$

assuming for instance that $x < M$ and the function f is supported by $[M, \infty)$.

To give the simplest, yet essential, example, we consider a positive number r and the various representations of the function,

$$F(z) = 1 + \frac{r}{z} = \frac{z+r}{z} = 1 - \int_{\mathbb{R}} \frac{r\,\mathrm{d}\delta_0(t)}{t-z} = \exp\left[-\int_{-r}^{0} \frac{\mathrm{d}t}{t-z}\right] = \det[(-r-z)(-z)^{-1}].$$

In this case, the underlying Hilbert space has dimension one and the two self-adjoint operators are $A = 0$ and $A - \xi\langle\cdot,\xi\rangle = -r$.

5.3.2. The reconstruction algorithm in one real variable. Returning to our main theme, and as a direct continuation of the previous section, we are interested in the exact reconstruction of the original $f : \mathbb{R} \longrightarrow [0, 1]$ from a finite set of its moments, or equivalently, from a Taylor polynomial of E_f at infinity. The algorithm described in this section is the diagonal Padé approximation of the exponential transform of the moment sequence. Its convergence, even beyond the real axis, is assured by a famous result discovered by A A Markov.

Let $a_0, a_1, \ldots, a_{2n}$ be a sequence of real numbers with the property that its exponential transform

$$\exp\left[\frac{1}{L}\left(\frac{a_0}{z} + \frac{a_1}{z^2} + \cdots \frac{a_{2n}}{z^{2n+1}}\right)\right] = 1 + \frac{b_0}{z} + \frac{b_1}{z^2} + \cdots$$

produces a non-negative Hankel matrix $(b_{k+l})_{k,l=0}^n$.

According to Markov's theorem, there exists at least one bounded self-adjoint operator $A \in L(H)$, with a cyclic vector ξ, such that,

$$\exp\left[\frac{1}{L}\left(\frac{a_0}{z} + \frac{a_1}{z^2} + \cdots + \frac{a_{2n}}{z^{2n+1}}\right)\right] = 1 + \frac{\langle \xi, \xi\rangle}{z} + \frac{\langle A\xi, \xi\rangle}{z^2} + \cdots + \frac{\langle A^{2n}\xi, \xi\rangle}{z^{2n+1}} + O\left(\frac{1}{z^{2n+2}}\right).$$

Let $k < n$ and H_k be the Hilbert subspace spanned by the vectors $\xi, A\xi, \ldots, A^{k-1}\xi$. Suppose that $\dim H_k = k$, which is equivalent to saying that $\det(b_{i+j})_{i,j=0}^{k-1} \neq 0$. Let π_k be the orthogonal projection of H onto H_k and let $A_k = \pi_k A \pi_k$. Then

$$\left\langle A_k^{i+j}\xi, \xi\right\rangle = \left\langle A_k^i\xi, A_k^j\xi\right\rangle = \langle A^i\xi, A^j\xi\rangle = \langle A^{i+j}\xi, \xi\rangle,$$

whenever $0 \leqslant i, j \leqslant k-1$. In other terms, for large values of $|z|$,

$$\langle (A-z)^{-1}\xi, \xi\rangle = \langle (A_k - z)^{-1}\xi, \xi\rangle + O\left(\frac{1}{z^{2k+1}}\right).$$

By construction, the vector ξ remains cyclic for the matrix $A_k \in L(H_k)$. Let $q_k(z)$ be the minimal polynomial of A_k, that is the monic polynomial of degree k which annihilates A_k. In particular,

$$q_k(z)\langle (A_k - z)^{-1}\xi, \xi\rangle = \langle (q_k(z) - q_k(A_k))(A_k - z)^{-1}\xi, \xi\rangle = p_{k-1}(z)$$

is a polynomial of degree $k-1$.

The two observations yield

$$q_k(z)\langle (A-z)^{-1}\xi, \xi\rangle = q_k(z)\langle (A_k - z)^{-1}\xi, \xi\rangle + O\left(\frac{1}{z^{k+1}}\right) = p_{k-1}(z) + O\left(\frac{1}{z^{k+1}}\right).$$

The resulting rational function $R_k(z) = \frac{p_{k-1}(z)}{q_k(z)}$ is characterized by the property

$$1 + \frac{b_0}{z} + \frac{b_1}{z^2} + \cdots = 1 + R_k(z) + O\left(\frac{1}{z^{2k+1}}\right);$$

it is known as the *Padé approximation* of order $(k-1, k)$, of the given series.

A basic observation is now in order: since $b_0, b_1, \ldots, b_{2k+1}$ is the power moment sequence of a positive measure, q_k is the associated orthogonal polynomial of degree k and p_k is a second order orthogonal polynomial of degree $k-1$. In particular their roots are simple and interlaced. We prove only the first assertion, the second one being of a similar nature. Indeed, let μ be the spectral measure of A localized at the vector ξ. Then, for $j < k$,

$$\int_{\mathbb{R}} t^j q_k(t)\,\mathrm{d}t = \langle A^j\xi, q_k(A)\xi\rangle = \left\langle A_k^j\xi, q_k(A_k)\xi\right\rangle = 0.$$

Assume now that we are in the extremal case $\det(b_{i+j})_{i,j=0}^n = 0$ and that n is the smallest integer with this property, that is $\det(b_{i+j})_{i,j=0}^{n-1} \neq 0$. Since

$$b_{i+j} = \langle A^i\xi, A^j\xi\rangle,$$

this means that the vectors $\xi, A\xi, \dots, A^n\xi$ are linearly dependent. Or equivalently that $H_n = H$ and consequently $A_n = A$.

According to the dictionary established above, this is another proof that the extremal case of the truncated moment 1-problem with data $a_0, a_1, \dots, a_{2n}$ admits a single solution. The unique function $f : \mathbb{R} \longrightarrow [0, 1]$ with this string of moments will then satisfy

$$\exp\left[-\int_{\mathbb{R}} \frac{f(t)\,\mathrm{d}t}{t-z}\right] = 1 + R_n(z) = 1 - \sum_{i=1}^{n} \frac{r_i}{a_i - z} = \det[(A - \xi\langle\cdot,\xi\rangle - z)(A-z)^{-1}]$$
$$= \prod_{i=1}^{n} \frac{b_i - z}{a_i - z},$$

where the spectrum of the matrix A is $\{a_1, \dots, a_n\}$, that of the perturbed matrix $B = A - \xi\langle\cdot,\xi\rangle$ is $b_1, \dots, b_n$ and r_i are positive numbers. Again, one can easily prove that $b_1 < a_1 < b_2 < a_2 < \cdots < b_n < a_n$. By the last example considered, we infer

$$f = \sum_{i=1}^{n} \chi_{[b_i, a_i]},$$

or equivalently

$$f = \frac{1}{2}\left[1 - \operatorname{sign}\frac{p_{k-1} + q_k}{q_k}\right].$$

The above computations can therefore be put into a (robust) reconstruction algorithm of all extremal functions f. The Hilbert space method outlined above has other benefits, too. We illustrate them with a proof of another celebrated result due to A A Markov, and related to the convergence of the mentioned algorithm, in the case of non-extremal functions.

Let μ be a positive measure, compactly supported on the real line and let $F(z) = \int_{\mathbb{R}}(t-z)^{-1}\,\mathrm{d}\mu(t)$ be its Cauchy transform. Then the diagonal Padé approximation $R_n(z) = p_{n-1}(z)/q_n(z)$ converges to $F(z)$ uniformly on compact subsets of $\mathbb{C}\backslash\mathbb{R}$.

This is the basic argument proving the statement: let A be the multiplication operator with the real variable on the Lebesgue space $H = L^2(\mu)$ and let $\xi = \mathbf{1}$ be its cyclic vector. The subspace generated by $\xi, A\xi, \dots, A^{n-1}\xi$ will be denoted as before by H_n and the corresponding compression of A by $A_n = \pi_n A \pi_n$.

If there exists an integer n such that $H = H_n$, then the discussion preceding the theorem shows that $F = R_n$ and we have nothing else to prove. Assume the contrary, that is the measure μ is not finite atomic.

Let $p(t)$ be a polynomial function, regarded as an element of H. Then

$$(A - A_n)p(t) = tp(t) - (\pi_n A \pi_n)p(t) = tp(t) - tp(t) = 0$$

provided that $\deg(p) < n$. Since $\|A_n\| \leqslant \|A\|$ for all n, and by the Weierstrass theorem, the polynomials are dense in H, we deduce

$$\lim_{n\to\infty} \|(A - A_n)h\| = 0, \qquad h \in H.$$

Fix a point $a \in \mathbb{C}\backslash\mathbb{R}$ and a vector $h \in H$. Then

$$\lim_{n\to\infty} \|[(A-a)^{-1} - (A_n - a)^{-1}]h\| = \lim_{n\to\infty} \|(A_n - a)^{-1}(A - A_n)(A-a)^{-1}h\|$$
$$\leqslant \lim_{n\to\infty} \frac{1}{|\mathrm{Im}\, a|}\|(A - A_n)(A-a)^{-1}h\| = 0.$$

A repeated use of the same argument shows that, for every $k \geqslant 0$,

$$\lim_{n\to\infty} \|[(A-a)^{-k} - (A_n - a)^{-k}]h\| = 0.$$

Choose a radius $r < |\mathrm{Im}\, a| \leqslant \|(A_n - a)^{-1}\|^{-1}$, so that the Neumann series

$$(A_n - z)^{-1} = (A_n - a - (z - a))^{-1} = \sum_{k=0}^{\infty} (z - a)^k (A_n - a)^{-k-1}$$

converges uniformly and absolutely, in n and z, in the disk $|z - a| \leqslant r$. Consequently, for a fixed vector $h \in H$,

$$\lim_{n\to\infty} \|(A_n - z)^{-1}h - (A - z)^{-1}h\| = 0,$$

uniformly in z, $|z - a| \leqslant r$. In particular,

$$\lim_{n\to\infty} R_n(z) = \langle (A_n - z)^{-1}\xi, \xi\rangle = \lim_{n\to\infty} \langle (A_n - z)^{-1}\xi, \xi\rangle$$
$$= \langle (A - z)^{-1}\xi, \xi\rangle = F(z),$$

uniformly in z, $|z - a| \leqslant r$.

Details and a generalization of the above operator theory approach to Markov theorem can be found in [130].

5.4. The exponential transform in two dimensions

We return now to two real dimensions, and establish an analog of the matrix model for Markov's moment problem. Fortunately this is possible due to the import of some key results in the theory of semi-normal operators. We expose first the analog of Markov's exponential transform, and second, we will make a digression into semi-normal operator theory, with the aim at realizing the exponential transform in terms of (infinite) matrices, and ultimately of reconstructing planar shapes from their moments.

The case of two real variables is special, partly due to the existence of a complex variable in $\mathbb{R}^2$. Let $g : \mathbb{C} \longrightarrow [0, 1]$ be a measurable function and let $\mathrm{d}A(\zeta)$ stand for the Lebesgue area measure. The *exponential transform* of g, is by definition the transform,

$$E_g(z) = \exp\left(-\frac{1}{\pi}\int_{\mathbb{C}} \frac{g(\zeta)\,\mathrm{d}A(\zeta)}{|\zeta - z|^2}\right), \qquad z \in \mathbb{C}\backslash\mathrm{supp}(g). \tag{180}$$

This expression invites to consider a polarization in z,

$$E_g(z, w) = \exp\left(-\frac{1}{\pi}\int_{\mathbb{C}} \frac{g(\zeta)\,\mathrm{d}A(\zeta)}{(\zeta - z)(\overline{\zeta} - \overline{w})}\right), \qquad z, w \in \mathbb{C}\backslash\mathrm{supp}(g). \tag{6.1}$$

The resulting function $E_g(z, w)$ is analytic in z and antianalytic in w, outside the support of the function g. Note that the integral converges for every pair $(z, w) \in \mathbb{C}^2$ except the diagonal $z = w$. Moreover, assuming by convention $\exp(-\infty) = 0$, a simple application of Fatou's theorem reveals that the function $E_g(z, w)$ extends to the whole $\mathbb{C}^2$ and it is separately continuous there. Details about these and other similar computations are contained in [127].

As before, the exponential transform contains, in its power expansion at infinity, the moments

$$a_{mn} = a_{mn}(g) = \int_{\mathbb{C}} z^m \overline{z}^n g(z)\,\mathrm{d}A(z), \qquad m, n \geqslant 0.$$

According to Riesz theorem these data determine g. We will denote the resulting series by

$$\exp\left[\frac{-1}{\pi}\sum_{m,n=0}^{\infty} \frac{a_{mn}}{z^{n+1}\overline{w}^{m+1}}\right] = 1 - \sum_{m,n=0}^{\infty} \frac{b_{mn}}{z^{n+1}\overline{w}^{m+1}}. \tag{181}$$

The exponential transform of a uniformly distributed mass on a disk is simple, and in some sense special, this being the building block for more complicated domains. A direct elementary computation leads to the following formulae for the unit disk **D**, cf [120],

$$E_{\mathbf{D}}(z,w)=\begin{cases}1-\dfrac{1}{z\overline{w}}, & z,w\in\overline{\mathbf{D}}^c,\\ 1-\dfrac{\overline{z}}{\overline{w}}, & z\in\mathbf{D},\ w\in\overline{\mathbf{D}}^c,\\ 1-\dfrac{w}{z}, & w\in\mathbf{D},\ z\in\overline{\mathbf{D}}^c,\\ \dfrac{|z-w|^2}{1-z\overline{w}}, & z,w\in\mathbf{D}.\end{cases}$$

Remark that $E_{\mathbf{D}}(z)=E_{\mathbf{D}}(z,z)$ is a rational function and its value for $|z|>1$ is $1-\frac{1}{|z|^2}$. The coefficients b_{mn} of the exponential transform are in this case particularly simple: $b_{00}=1$ and all other values are zero.

Once more, an additional structure of the exponential transform in two variables comes from operator theory. More specifically, for every measurable function $g:\mathbb{C}\to[0,1]$ of compact support there exists a unique irreducible, linear bounded operator $T\in L(H)$ acting on a Hilbert space H, with rank-1 self-commutator $[T^*,T]=\xi\otimes\xi=\xi\langle\cdot,\xi\rangle$, which factors E_g as follows:

$$E_g(z,w)=1-\langle(T^*-\overline{w})^{-1}\xi,(T^*-\overline{z})^{-1}\xi\rangle,\qquad z,w\in\operatorname{supp}(g)^c. \quad (182)$$

As a matter of fact, with a proper extension of the definition of localized resolvent $(T^*-\overline{w})^{-1}\xi$ the above formula makes sense on the whole $\mathbb{C}^2$. The function g is called the *principal function* of the operator T. The following section will contain a brief incursion into this territory of operator theory.

Let $g:\mathbb{C}\to[0,1]$ be a measurable function and let $E_g(z,w)$ be its polarized exponential transform. We retain from the above discussion the fact that the kernel,

$$1-E_g(z,w),\qquad z,w\in\mathbb{C},$$

is positive definite. Therefore the distribution $H_g(z,w)=-\frac{\partial}{\partial\overline{z}}\frac{\partial}{\partial w}E_g(z,w)$ has compact support and it is positive definite, in the sense,

$$\int_{\mathbb{C}^2}H_g(z,w)\phi(z)\overline{\phi(w)}\,\mathrm{d}A(z)\,\mathrm{d}A(w)\geqslant 0,\qquad \phi\in C^\infty(\mathbb{C}).$$

If g is the characteristic function of a bounded domain $\Omega\subset\mathbb{C}$, then it is elementary to see that the distribution $H_\Omega(z,w)=H_g(z,w)$ is given on $\Omega\times\Omega$ by a smooth, jointly integrable function which is analytic in $z\in\Omega$ and antianalytic in $w\in\Omega$, see [120].

In particular, this gives the useful representation,

$$E_\Omega(z,w)=1-\frac{1}{\pi^2}\int_{\Omega^2}\frac{H_\Omega(u,v)\,\mathrm{d}A(u)\,\mathrm{d}A(v)}{(u-z)(\overline{v}-\overline{w})},\qquad z,w\in\overline{\Omega}^c,$$

where the kernel H_Ω is positive definite in $\Omega\times\Omega$.

The example of the disk considered in this section suggests that the exterior exponential transform of a bounded domain $E_\Omega(z,w)$ may extend analytically in each variable inside Ω. This is true whenever $\partial\Omega$ is real analytic smooth. In this case there exists an analytic function S defined in a neighborhood of $\partial\Omega$, with the property

$$S(z)=\overline{z},\qquad z\in\partial\Omega.$$

The anticonformal local reflection with respect to $\partial\Omega$ is then the map $z\mapsto\overline{S(z)}$; for this reason $S(z)$ is called the *Schwarz function* of the real analytic curve $\partial\Omega$, introduced earlier in this text. Let ω be a relatively compact subdomain of Ω, with smooth boundary, too, and such that the Schwarz function $S(z)$ is defined on a neighborhood of $\Omega\backslash\omega$. A formal use of Stokes' theorem yields

$$\begin{aligned} 1 - E_\Omega(z,w) &= \frac{1}{4\pi^2}\int_{\partial\Omega}\int_{\partial\Omega} H_\Omega(u,v)\frac{\overline{u}du}{u-z}\frac{v\,\mathrm{d}\overline{v}}{\overline{v}-\overline{w}} \\ &= \frac{1}{4\pi^2}\int_{\partial\omega}\int_{\partial\omega} H_\Omega(u,v)\frac{\overline{u}du}{u-z}\frac{v\,\mathrm{d}\overline{v}}{\overline{v}-\overline{w}}. \end{aligned}$$

But the latter integral is analytic/antianalytic for $z, w \in \overline{\omega}^c$. A little more work with the above Cauchy integrals leads to the following remarkable formula for the analytic extension of $E_\Omega(z,w)$ from $z, w \in \overline{\Omega}^c$ to $z, w \in \overline{\omega}^c$,

$$F(z,w) = \begin{cases} E(z,w), & z, w \in \Omega^c, \\ (z - \overline{S(w)})(S(z) - \overline{w})H_\Omega(z,w), & z, w \in \Omega\backslash\overline{\omega}. \end{cases}$$

The study outlined above of the analytic continuation phenomenon of the exponential transform $E_\Omega(z,w)$ led to a proof of *a priori* regularity of boundaries of domains which admit analytic continuation of their Cauchy transform. The most general result of this type was obtained by different means by Sakai. We simply state the result, giving in this way a little more insight into the proof of the regularity of the boundaries of quadrature domains.

Let Ω be a bounded planar domain with the property that its Cauchy transform

$$\hat{\chi}_\Omega(z) = \frac{-1}{\pi}\int_\Omega \frac{\mathrm{d}A(w)}{w-z}, \qquad z \in \overline{\Omega}^c$$

extends analytically across $\partial\Omega$. Then the boundary $\partial\Omega$ is real analytic.

Moreover, Sakai has classified the possible singular points of the boundary of such a domain. For instance angles not equal to 0 or π cannot occur on the boundary.

5.5. Semi-normal operators

A normal operator is modeled via the spectral theorem as multiplication by the complex variable on a vector-valued Lebesgue L^2-space. The interplay between measure theory and the structure of normal operators is well known and widely used in applications. One step further, there are by now well-understood functional models, and a complete classification for classes of close to normal operators. We record below a few aspects of the theory of semi-normal operators with trace-class self-commutators. They will be serve as Hilbert space counterparts for the study of moving boundaries in two dimensions. The reader is advised to consult the monographs [127, 131] for full details.

Let H be a separable, complex Hilbert space and let $T \in \mathcal{L}(H)$ be a linear bounded operator. We assume that the self-commutator $[T^*, T] = T^*T - TT^*$ is trace-class, and call T semi-normal. If $[T^*, T] \geqslant 0$, then T is called *hypo-normal*. For a pair of polynomials $p(z,\overline{z}), q(z,\overline{z})$ one can choose (at random) an ordering in the functional calculus $p(T,T^*), q(T,T^*)$, for instance putting all adjoints to the left of all other monomials. The functional

$$(p,q) \to \mathrm{trace}[p(T,T^*), q(T,T^*)]$$

is then well defined, independent of the ordering in the functional calculus and possesses the algebraic identities of the Jacobian $\frac{\partial(p,q)}{\partial(\overline{z},z)}$. A direct (algebraic) reasoning will imply the existence of a distribution $u_T \in \mathcal{D}'(\mathbb{C})$ satisfying

$$\mathrm{trace}[p(T,T^*), q(T,T^*)] = u_T\left[\frac{\partial(p,q)}{\partial(\overline{z},z)}\right],$$

see [132]. The distribution u_T exists in any number of variables (that is for tuples of self-adjoint operators subject to a trace-class multi-commutator condition) and it is known as the *Helton–Howe functional*.

Dimension two is special because of a theorem of J D Pincus which asserts that $u_T = \frac{1}{\pi} g_T \, \mathrm{dA}$, that is u_T is given by an integrable function g_T, called the *principal function* of the operator T, see [133, 134].

The analogy between the principal function and the phase shift (the density of the measure appearing in Markov's moment problem in one variable) is worth mentioning in more detail. More precisely, if $B = A - K$ is a trace-class, self-adjoint perturbation of a bounded self-adjoint operator $A \in L(H)$, then for every polynomial $p(z)$, Krein's *trace formula* holds

$$\mathrm{tr}[p(B) - p(A)] = \int_{\mathbb{R}} p'(t) f_{A,B}(t) \, \mathrm{d}t,$$

where $f_{A,B}$ is the corresponding phase-shift function, [128]. It is exactly this link between Hilbert space operations and functional expressions which bring the two scenarios very close. Taking one step further, exactly as in the one variable case, the moments of the principal function can be interpreted in terms of the Hilbert space realization, as follows:

$$mk \int z^{m-1} \overline{z}^{k-1} g_T(z) \, \mathrm{dA} = \mathrm{trace}[T^{*k}, T^m], \qquad k, m \geqslant 1.$$

In general, the principal function can be regarded as a generalized Fredholm index of T, that is, when the left-hand side below is well defined, we have

$$\mathrm{ind}(T - \lambda) = -g_T(\lambda).$$

Moreover g_T enjoys the functoriality properties of the index, and it is obviously invariant under trace-class perturbations of T. Moreover, in the case of a fully non-normal operator T,

$$\mathrm{supp}\, g_T = \sigma(T),$$

and various parts of the spectrum $\sigma(T)$ can be interpreted in terms of the behavior of g_T, see for details [127].

To give a simple, yet non-trivial, example we proceed as follows. Let Ω be a planar domain bounded by a smooth Jordan curve Γ. Let $H^2(\Gamma)$ be the closure of complex polynomials in the space $L^2(\Gamma, \mathrm{d}s)$, where $\mathrm{d}s$ stands for the arc length measure along Γ (the so-called Hardy space attached to Γ). The elements of $H^2(\Gamma)$ extend analytically to Ω. The multiplication operator by the complex variable, $T_z f = zf$, $f \in H^2(\Gamma)$, is obviously linear and bounded. The regularity assumption on Γ implies that the commutator $[T_z, T_z^*]$ is trace class. Moreover, the associated principal function is the characteristic function of Ω, so that the trace formula above becomes

$$\mathrm{trace}[p(T_z, T_z^*), q(T_z, T_z^*)] = \frac{1}{\pi} \int_\Omega \frac{\partial(p, q)}{\partial(\overline{z}, z)} \, \mathrm{dA}, \qquad p, q \in \mathbb{C}[z, \overline{z}].$$

See for details [127, 131].

A second, more interesting (generic example this time) can be constructed as follows. Let $u(t), v(t)$ be real-valued, bounded continuous functions on the interval [0, 1]. Consider the singular integral operator, acting on the Lebesgue space $L^2([0, 1], \mathrm{d}t)$ by the formula

$$(Tf)(t) = tf(t) + \mathrm{i}[u(t)f(t)] + \frac{1}{\pi} \int_0^1 \frac{v(t)v(s)f(s)\,\mathrm{d}s}{s - t}.$$

Then it is easy to see that the self-commutator $[T^*, T]$ is rank 1. The principal function g_T will be in this case the characteristic function of the closure of the domain G given by the constraints

$$G = \{(x, y) \in \mathbb{R}^2; |y - u(x)| \leqslant v(x)^2, x \in [0, 1]\}.$$

Based on a refinement of this example, in general every *hyponormal operator* with trace-class self-commutator can be represented by such a singular integral model, with matrix-valued functions u, v, acting on a direct integral of Hilbert spaces over [0, 1]; in which case the principal function relates directly to Krein's phase shift, by the following remarkable formula due to Pincus [133]:

$$g_T(x, y) = f_{u(x)-v(x)^*v(x),u(x)+v(x)^*v(x)}(y).$$

The case of rank-1 self-commutators is singled out in the following key classification result:

There exists a bijective correspondence $T \mapsto g_T$ between irreducible hyponormal operators T, with rank-1 self-commutator, and bounded measurable functions with compact support in the complex plane.

An invariant formula, relating the moments of the principal function g to the Hilbert space operator T, $[T^*, T] = \xi\langle\cdot, \xi\rangle$,, satisfying $g_T = g$, a.e. is furnished by the determinantal formula

$$\exp\left(-\frac{1}{\pi}\int_{\mathbb{C}}\frac{g(\zeta)\,\mathrm{d}A(\zeta)}{(\zeta - z)(\overline{\zeta} - \overline{w})}\right) = \det[(T^* - \overline{w})^{-1}(T - z)(T^* - \overline{w})(T - z)^{-1}]$$
$$= 1 - \langle (T^* - \overline{w})^{-1}\xi, (T^* - \overline{z})^{-1}\xi\rangle, \qquad z, w \in \operatorname{supp}(g)^c.$$

This formula explains the positivity property of the exponential transform, alluded to in the previous section.

The bijective correspondence between classes $g \in L^\infty_{\mathrm{comp}}(\mathbb{C})$, $0 \leqslant g \leqslant 1$ and irreducible operators T with rank-1 self-commutator was exploited in [135, 136] for solving the L-problem of moments in two variables. The theory of the principal function has inspired and played a basic role in the foundations of modern non-commutative geometry (specifically the cyclic cohomology of operator algebras) and non-commutative probability.

We have to stress the fact that the above bijective correspondence between 'shade functions' g_T and irreducible hyponormal operators T with rank-1 self-commutator can in principle transfer *any* dynamic $g(t)$ into a Hilbert space operator dynamic $T(t)$. However, the details of the evolution law of $T(t)$ even in the case of elliptic growth are not trivial, nor make the integration simpler. We will see some relevant low-degree examples in the following section.

5.5.1. Applications: Laplacian growth. To give a single abstract illustration, consider a growing family of bounded planar domains $D(t)$ with smooth boundary,

$$D(t) \subset D(s), \qquad \text{whenever} \quad t < s.$$

The evolution of the exponential transforms

$$E_{D(t)}(z, w) = \exp\left[\frac{-1}{\pi}\int_{D(t)}\frac{\mathrm{d}A(\zeta)}{(\zeta - z)(\bar{\zeta} - \bar{w})}\right]$$

is governed by the differential equation (in the standard vector calculus notation)

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{D(t)}(z, w) = \frac{-1}{\pi}E_{D(t)}(z, w)\int_{\partial D(t)}\frac{V_n\,\mathrm{d}\ell(\zeta)}{(\zeta - z)(\overline{\zeta} - \overline{w})}.$$

Any evolution law at the level of the pair $(T(t), \xi(t))$ will have the form

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{D(t)}(z, w) = \langle (T^*(t) - \overline{w})^{-1})T^*(t)(T^*(t) - \overline{w})^{-1})\xi(t), (T^*(t) - \overline{z})^{-1}\xi(t)\rangle$$
$$- \langle (T^*(t) - \overline{w})^{-1}\xi'(t), (T^*(t) - \overline{z})^{-1}\xi(t)\rangle$$
$$+ \langle (T^*(t) - \overline{w})^{-1}\xi(t), (T^*(t) - \overline{z})^{-1}T^*(t)(T^*(t) - \overline{z})^{-1}\xi(t)\rangle$$
$$- \langle (T^*(t) - \overline{w})^{-1})\xi(t), (T^*(t) - \overline{z})^{-1})\xi'(t)\rangle.$$

A series of simplification in the case of elliptic growth are immediate: for instance $\|\xi(t)\|$ is proportional to the area of $D(t)$, whence we can choose the vector of the form

$$\xi(t) = t\xi(0).$$

Second, the higher harmonic moments are preserved by the evolution, whence the Cauchy transform/resolvent

$$\frac{\mathrm{d}}{\mathrm{d}t}\pi\langle\xi(t), (T^*(t) - \overline{z})^{-1}\xi(t)\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\int_{D(t)} \frac{\mathrm{d}A(\zeta)}{\zeta - z} = -\frac{c}{z}$$

gives full information about the first row and first column in the matrix representation of $T^*(t)$ in the basis obtained by orthonormalizing the sequence $\xi(t), T^*(t)\xi(t), T^{*2}\xi(t), \ldots$. The reader can consult the paper [122] for more details about computations related to the above ones.

5.6. *Linear analysis of quadrature domains*

If we would infer from the one-variable picture a good class of extremal domains for Markov's L-problem in two variables we would choose the disjoint unions of disks, as immediate analogs of disjoint unions of intervals. In reality, the nature of the complex plane is much more complicated, but again, fortunately for our survey, the class of quadrature domains plays the role of extremal solutions in two real dimensions.

Recall from our previous sections that a bounded domain Ω of the complex plane is called a *quadrature domain* (always henceforth for analytic functions) if there exists a finite set of points $a_1, a_2, \ldots, a_d \in \Omega$, and real weights $c_1, c_2, \ldots, c_d$, with the property

$$\int_\Omega f(z)\,\mathrm{d}A(z) = c_1 f(a_1) + c_2 f(a_2) + \cdots + c_d f(a_d), \qquad f \in AL^1(\Omega),$$

where the latter denotes the space of all integrable analytic functions in Ω. In case some of the above points coincide, a derivative of f can correspondingly be evaluated.

Let Ω be a bounded planar domain with moments

$$a_{mn} = a_{mn}(\Omega) = \int_\Omega z^m \overline{z}^n\,\mathrm{d}A(z), \qquad m, n \geqslant 0.$$

The exponential transform produces the sequence of numbers $b_{mn} = b_{mn}(\Omega), m, n \geqslant 0$. Let T denote the irreducible hyponormal operator with rank-1 self-commutator $[T^*, T] = \xi\langle\cdot, \xi\rangle$. In virtue of the factorization (182),

$$b_{mn} = \langle T^{*m}\xi, T^{*n}\xi\rangle, \qquad m, n \geqslant 0.$$

Hence the matrix $(b_{mn})_{m,n=0}^\infty$ turns out to be non-negative definite. The following result identifies a part of the extremal solutions of the L-problem of moments as the class of quadrature domains:

A bounded planar domain Ω is a quadrature domain if and only if there exists a positive integer $d \geqslant 1$ with the property $\det(b_{mn}(\Omega))_{m,n=0}^d = 0$.

For a proof see [135]. The vanishing condition in the statement is equivalent to the fact that the span H_d of the vectors $\xi, T^*\xi, T^{*2}\xi, \ldots$ is finite-dimensional (in the Hilbert space where the associated hyponormal operator T acts). Thus, if Ω is a quadrature domain with the corresponding hyponormal operator T, and T_d is the compression of T to the d-dimensional subspace H_d, then

$$E_\Omega(z, w) = 1 - \langle(T_d^* - \overline{w})^{-1}\xi, (T_d^* - \overline{z})^{-1}\xi\rangle, \qquad z, w \in \overline{\Omega}^c.$$

In particular, this proves that the exponential transform of a quadrature domain is a rational function. As a matter of fact a more precise statement can easily be deduced:

Let Ω be the quadrature domain defined above. Then

$$E_\Omega(z, w) = \frac{Q(z, w)}{P(z)\overline{P(w)}}, \qquad z, w \in \overline{\Omega}^c.$$

This result offers an efficient characterization of quadrature domains in terms of a finite set of their moments (see the reconstruction section below) and it opens a natural correspondence between quadrature domains and certain classes of finite rank matrices. We only describe a few results in this direction. For more details see [120, 122, 135].

In the conditions of the above result, let Ω be a quadrature domain with associated hyponormal operator T; let $H_0 = \bigvee_{k\geqslant 0} T^{*k}\xi$ and let p denote the orthogonal projection of the Hilbert space H (where T acts) onto H_0. Denote $C_0 = pTp$ (the compression of T to the d-dimensional space H_0) and $D_0^2 = [T^*, T]$. Then the operator T has a two block-diagonal structure

$$T = \begin{pmatrix} C_0 & 0 & 0 & 0 & \dots \\ D_1 & C_1 & 0 & 0 & \dots \\ 0 & D_2 & C_2 & 0 & \dots \\ 0 & 0 & D_3 & C_3 & \dots \\ \vdots & & \vdots & & \ddots \end{pmatrix},$$

where the entries are all $d \times d$ matrices, recurrently defined by the system of equations

$$\begin{cases} [C_k{}^*, C_k] + D_{k+1}{}^* D_{k+1} = D_k D_k{}^* \\ C_{k+1}{}^* D_{k+1} = D_{k+1} C_k{}^*, \qquad k \geqslant 0. \end{cases}$$

Note that $D_k > 0$ for all k. This decomposition has an array of consequences:

(i) The spectrum of C_0 coincides with the quadrature nodes of Ω;
(ii) $\Omega = \{z;\ \|(C_0^* - \overline{z})^{-1}\xi\| > 1\}$ (up to a finite set);
(iii) The quadrature identity becomes

$$\int_\Omega f(z)\,\mathrm{dA}(z) = \pi\langle f(C_0)\xi, \xi\rangle,$$

for f analytic in a neighborhood of $\overline{\Omega}$;
(iv) The Schwarz function of Ω is

$$S(z) = \overline{z} - \langle \xi, (C_0^* - \overline{z})^{-1}\xi\rangle + \langle \xi, (T^* - \overline{z})^{-1}\xi\rangle,$$

where $z \in \Omega$.

To give the simplest and most important example, let $\Omega = \mathbf{D}$ be the unit disk (which is a quadrature domain of order one). Then the associated operator is the unilateral shift $T = T_z$ acting on the Hardy space $H^2(\partial\mathbf{D})$. Denoting by z^n the orthonormal basis of this space we have $Tz^n = z^{n+1}, n \geqslant 0$, and $[T^*, T] = 1\langle\cdot, 1\rangle$ is the projection onto the first coordinate $1 = z^0$. The space H_0 is one dimensional and $C_0 = 0$. This will propagate to $C_k = 0$ and $D_k = 1$ for all k. Thus the matricial decomposition of T becomes the familiar realization of the shift as an infinite Jordan block.

In view of the linear algebra realization outlined in the preceding section we obtain more information about the defining equation of the quadrature domain. For instance,

$$\frac{Q(z, \overline{w})}{P(z)\overline{P(w)}} = 1 - \langle (C_0^* - \overline{w})^{-1}\xi, (C_0^* - \overline{z})^{-1}\xi\rangle,$$

which yields

$$Q(z, z) = |P(z)|^2 - \sum_{k=0}^{d-1} |Q_k(z)|^2,$$

where Q_k is a polynomial of degree k in z, see [122].

Thus the exponential transform of a quadrature domain contains explicitly the irreducible polynomial Q which defines the boundary and the polynomial P which vanishes at the quadrature nodes. By putting together all these remarks we obtain a strikingly similar picture to that of a single variable. More specifically, if Ω is a quadrature domain with d nodes, as given above, and associated hyponormal operator T, then

$$\begin{aligned} E_\Omega(z, w) &= \frac{Q(z, w)}{P(z)\overline{P(w)}} = 1 - \langle (T_d^* - \overline{w})^{-1}\xi, (T_d^* - \overline{z})^{-1}\xi \rangle \\ &= \frac{1}{\pi^2} \sum_{i,j=1}^{d} H_\Omega(a_i, a_j) \frac{c_i}{a_i - z} \frac{\overline{c_j}}{\overline{a_j} - \overline{w}}, \qquad z, w \in \overline{\Omega}^c. \end{aligned}$$

In particular we infer, assuming that all nodes are simple,

$$-\pi^2 \frac{Q(a_i, a_j)}{P'(a_i)\overline{P'(a_j)}} = c_i \overline{c_j} H_\Omega(a_i, \overline{a_j}), \qquad 1 \leqslant i, j \leqslant d.$$

For details see [135, 122].

The interplay between these additive, multiplicative and Hilbert-space decompositions of the exponential transform gives an exact reconstruction algorithm of a quadrature domain from its moments. The following section will be devoted to this algorithm.

Before ending the present section we consider an illustration of the above formulae. Let $\Omega = \cup_{i=1}^{d} D(a_i, r_i)$ be a union of d pairwise disjoint disks. This is a quadrature domain with data

$$\begin{aligned} P(z) &= (z - a_1) \cdots (z - a_d), \\ Q(z, w) &= \left[(z - a_1)(\overline{w} - \overline{a_1}) - r_1^2\right] \cdots \left[(z - a_d)(\overline{w} - \overline{a_d}) - r_d^2\right]. \end{aligned}$$

The associated matrix T_d is also computable, involving a sequence of square roots of matrices, but we do not need here its precise form. Whence the exponential transform is, for large values of $|z|, |w|$,

$$E_\Omega(z, w) = \prod_{i=1}^{d} \left[1 - \frac{r_i^2}{(z - a_i)(\overline{w} - \overline{a_i})}\right] = 1 + \sum_{i,j=1}^{d} \frac{Q(a_i, \overline{a_j})}{P'(a_i)\overline{P'(a_j)}} \frac{r_i}{a_i - z} \frac{r_j}{\overline{a_j} - \overline{w}}.$$

The essential positive definiteness of the exponential transform of an arbitrary domain can be deduced, via an approximation argument, from the positivity of the matrix $(-Q(a_i, \overline{a_j}))_{i,j=1}^d$, where Q is the defining equation of a disjoint union of disks. We note that $(-Q(a_i, \overline{a_j}))_{i,j=1}^d \geqslant 0$ is only a necessary condition for the disks $D(a_i, r_i)$, $1 \leqslant i \leqslant d$, to be disjoint. Exact computations for $d = 2$ immediately show that this matrix can remain positive definite even the two disks overlap a little. However, if two disks overlap, then, by adding an external disk, even far away, this prevents the new 3×3 matrix to be positive definite.

We end this section with two examples, covering the totality of quadrature domains of order two.

Quadrature domains with a double node. Let $z = w^2 + bw$ be the conformal mapping of the disk $|w| < 1$, where $b \geqslant 2$. Then z describes a quadrature domain Ω of order 2, whose boundary has the equation

$$Q(z, \overline{z}) = |z|^4 - (2 + b^2)|z|^2 - b^2 z - b^2 \overline{z} + 1 - b^2 = 0.$$

The Schwarz function of Ω has a double pole at $z = 0$, whence the associated 2×2-matrix C_0 is nilpotent. Moreover, we know that

$$|z|^4 \|(C_0^* - \overline{z})^{-1}\xi\|^2 = |z|^4 - P(z, \overline{z}).$$

Therefore

$$\|(C_0^* + \overline{z})\xi\|^2 = (2 + b^2)|z|^2 + b^2 z + b^2\overline{z} + b^2 - 1,$$

or equivalently: $\|\xi\|^2 = 2 + b^2$, $\langle C_0^*\xi, \xi\rangle = b^2$ and $\|C_0^*\xi\|^2 = b^2 - 1$.

Consequently the linear data of the quadrature domain Ω are

$$C_0^* = \begin{pmatrix} 0 & \frac{b^2-1}{(b^2-2)^{1/2}} \\ 0 & 0 \end{pmatrix}, \qquad \xi = \begin{pmatrix} \frac{b^2}{(b^2-1)^{1/2}} \\ \left(\frac{b^2-2}{b^2-1}\right)^{1/2} \end{pmatrix}.$$

Quadrature domains with two distinct nodes. Assume that the nodes are fixed at ± 1. Hence $P(z) = z^2 - 1$. The defining equation of the quadrature domain Ω of order two with these nodes is

$$Q(z, \overline{z}) = (|z + 1|^2 - r^2)(|z - 1|^2 - r^2) - c,$$

where r is a positive constant and $c \geqslant 0$ is chosen so that either Ω is a union of two disjoint open disks (in which case $c = 0$), or $Q(0, 0) = 0$, see [109]. A short computation yields

$$Q(z, \overline{z}) = z^2\overline{z}^2 - 2rz\overline{z} - z^2 - \overline{z}^2 + \alpha(r),$$

where

$$\alpha(r) = \begin{cases} (1 - r^2)^2, & r < 1 \\ 0, & r \geqslant 1. \end{cases}$$

One step further, we can identify the linear data from the identity

$$|P(z)|^2(1 - \|(C_0^* - \overline{z})^{-1}\xi\|^2) = Q(z, \overline{z}). \tag{183}$$

Consequently,

$$\xi = \begin{pmatrix} \sqrt{2r} \\ 0 \end{pmatrix}, \qquad C_0^* = \begin{pmatrix} 0 & \frac{\sqrt{2r}}{\sqrt{1-\alpha(r)}} \\ \frac{\sqrt{1-\alpha(r)}}{\sqrt{2r}} & 0 \end{pmatrix}.$$

This simple computation illustrates the fact that, although the process is affine in r, the linear data of the growing domains have discontinuous derivatives at the exact moment when the connectivity changes.

5.7. Signed measures, instability, uniqueness

Contrary to the uniqueness of a quadrature domain for subharmonic functions with a prescribed quadrature measure, quadrature domains for harmonic or analytic functions are not determined by the quadrature nodes and weights. This is an intriguing global phenomenon which has haunted mathematicians for many decades. We briefly record below some significant discoveries in this direction.

Consider quadrature domains for harmonic test functions and real-valued measures (174). As to the relationship between the geometry of Ω and the location of supp μ there are then drastic differences between the cases of having all $c_j > 0$, respectively, having no restrictions on the signs of c_j. This is clearly demonstrated in the following theorem due to Sakai [137, 138]. The second part of the theorem is discussed (and proved) in some other forms also in [106–108, 139, 140], for example:

Let r and R be positive numbers, $R \geqslant 2r$. Consider measures μ of the form (174) with c_j real and related to r and R by

$$\operatorname{supp}\mu \subset B(0, r), \tag{184}$$

$$\sum_{j=1}^{n} c_j = \pi R^2. \tag{185}$$

(i) *If $\mu \geqslant 0$, then any quadrature domain Ω for harmonic functions for μ is also a quadrature domain for subharmonic functions. Hence the previous result applies, and in addition*

$$B(0, R - r) \subset \Omega \subset B(0, R + r).$$

(ii) *With μ not necessarily $\geqslant 0$, and with no restrictions on $\sum_{j=1}^{n} |c_j|$ and n, any bounded domain containing $B(0, r)$ and having area πR^2 can be uniformly approximated by quadrature domains for harmonic functions for measures μ satisfying (184), (185).*

With μ a signed measure of the form (174) we still have $\sum_{j=1}^{n} c_j = |\Omega|$, but $\sum_{j=1}^{n} |c_j|$ may be much larger. In view of the theorem, the ratio

$$\rho = \frac{\sum_{j=1}^{n} c_j}{\sum_{j=1}^{n} |c_j|} = \frac{\int \mathrm{d}\mu}{\int |\mathrm{d}\mu|}$$

($0 < \rho \leqslant 1$) might give an indication of how strong is the coupling between the geometry of $\operatorname{supp}\mu$ and the geometry of Ω.

As mentioned, a quadrature domain for harmonic functions is not always uniquely determined by its measure μ. Still there is uniqueness at the infinitesimal level: if

$$\sum_{j=1}^{n} c_j \varphi(a_j) = \int_{\Omega} \varphi \, \mathrm{dA} \tag{186}$$

and (for example) the a_j are kept fixed, then one can always increase the c_j (indefinitely) and get a unique evolution of Ω (Hele–Shaw evolution). If $\partial\Omega$ has no singularities then one can also decrease the c_j slightly and have a unique evolution (backward Hele–Shaw, which is ill-posed). Thus it makes sense to write

$$\Omega = \Omega(c_1, \ldots, c_n)$$

for c_j in some interval around the original values. Note however that decreasing the c_j makes the ratio ρ decrease, indicating a loss of control or stability.

In the simply connected case, Ω will be the image of the unit disc $\mathbf{D}$ under a rational conformal map $f = f_{(c_1,\ldots,c_n)} : \mathbf{D} \to \Omega(c_1, \ldots, c_n)$. This rational function is simply the conformal pull-back of the meromorphic function $(z, S(z))$ on the Schottky double of Ω to the Schottky double of $\mathbf{D}$, the latter being identified with the Riemann sphere. It follows that the poles of f are the mirror points (with respect to the unit circle) of the points $f^{-1}(a_j)$. When the c_j increase then the $|f^{-1}(a_j)|$ decrease (this follows by an application of Schwarz' lemma to $f^{-1}_{\text{larger}\, c_j} \circ f_{\text{original}\, c_j}$), hence the poles of f move away from the unit circle. Conversely, the poles of f approach the unit circle as the c_j decrease, also indicating a loss of stability.

For decreasing c_j the evolution $\Omega(c_1, \ldots, c_n)$ always breaks down by singularity development of $\partial\Omega$ or $\partial\Omega$ reaching some of the points a_j (see, e.g., [103, 141]) before Ω is empty, except in the case that $\Omega(c_1, \ldots, c_n)$ is a quadrature domain for subharmonic functions. In the latter case the c_j (necessarily positive) can be decreased down to zero, and Ω will be empty in the limit $c_1 = \cdots = c_n = 0$. However, it may happen that $\Omega(c_1, \ldots, c_n)$ breaks up into components under the evolution.

Assume now that Ω is simply connected. Then the analytic and harmonic functions are equivalent as test classes for (186). In the limit case that all the points a_j coincide, say $a_1 = \cdots = a_n = 0$, then (186) corresponds to

$$\sum_{j=1}^{n} c_j \varphi^{(j-1)}(0) = \int_\Omega \varphi \, \mathrm{dA} \tag{187}$$

for φ analytic. The c_j (allowed to be complex) now have a slightly different meaning than before. In fact, they are essentially the analytic moments of Ω,

$$c_j = \frac{1}{(j-1)!} \int_\Omega z^{j-1} \, \mathrm{d}A \qquad (j = 1, \ldots, n).$$

The higher order moments vanish, and the conformal map $f = f_{(c_1,\ldots,c_n)} : \mathbf{D} \to \Omega(c_1, \ldots, c_n)$ (normalized by $f(0) = 0$, $f'(0) > 0$) is a polynomial of degree n. A precise form of the local bijectivity of the map $(c_1, \ldots, c_n) \mapsto \Omega(c_1, \ldots, c_n)$ has been established by Kouznetsova and Tkachev [142, 143], who proved an explicit formula for the (nonzero) Jacobi determinant of the map from the coefficients of f to the moments $(c_1, \ldots, c_n)$). This formula was conjectured (and proved in some special cases) by Ullemar [144].

On the global level, it does not seem to be known whether (187), or (186), with a given left member, can hold for two different simply connected domains and all analytic φ.

Leaving the realm of quadrature domains, an explicit example of two different simply connected domains having the same analytic moments has been given by Sakai [145]. The idea of the example is that a disc and a concentric annulus of the same area have equal moments. If the disc and annulus are not concentric, then the union of them (if disjoint) will have the same moments as the domain obtained by interchanging their roles. Arranging everything carefully, with removing and adding some common parts, two different Jordan domains having equal analytic moments can be obtained. Similar examples were known earlier by Celmins [146], and probably even by P S Novikov. On the positive side, a classical theorem of Novikov [147] asserts that domains which are starshaped with respect to one and the same point are uniquely determined by their moments. See [148] for further discussions.

Returning now to quadrature domains, there is definitely no uniqueness for harmonic and analytic test classes if multiply connected domains are allowed. If Ω has connectivity $m + 1$ ($m \geqslant 1$), i.e., has m 'holes', then there is generically an m-parameter family $\Omega(t_1, \ldots, t_m)$ of domains such that $\Omega(0, \ldots, 0) = \Omega$ and

$$\frac{\partial}{\partial t_j} \int_{\Omega(t_1,\ldots,t_m)} \varphi \, \mathrm{dA} = 0 \qquad (j = 1, \ldots, m)$$

for every φ analytic in a neighborhood of the domains. These deformations are Hele–Shaw evolutions, driven not by Green functions but by 'harmonic measures', i.e., regular harmonic functions which take (different) constant boundary values on the components of $\partial\Omega$.

It follows that multiply connected quadrature domains for analytic functions for a given μ occur in continuous families. It even turns out [48, 149] that *any two* algebraic domains for the same μ can be deformed into each other through families as above. Thus there is a kind of uniqueness at a higher level: given any μ there is at most one connected family of algebraic domains belonging to it.

For harmonic quadrature domains there are no such continuous families (choosing $\varphi(z) = \log|z - a|$ in (186) with $a \in \mathbb{C}\backslash\overline{\Omega}$ in the holes stops them), but one can still construct examples with a discrete set of different domains for the same μ. It is for example possible to imitate the example with a disc and an annulus with quadrature domains for measures μ of the form (174), with $a_j = \mathrm{e}^{2\pi j/n}$ ($n \geqslant 3$) and $c_1 = \cdots = c_n = c > 0$ suitably chosen.

However, it seems very difficult to imitate the full Sakai construction, with 'removing and adding some common parts', in the context of quadrature domains. Therefore it is not at all easy to construct different simply connected quadrature domains for the same μ.

We end this section with the simplest example of a continuous class of quadrature domains with the same quadrature data.

Three points, non-simply connected quadrature domains and the non-uniqueness phenomenon. Quadrature domains (for analytic functions) with at most two nodes, as in the above examples, are uniquely determined by their quadrature data and are simply connected. For three nodes and more it is no longer so. The following example, taken from [109], with three nodes and symmetry under rotations by $2\pi/3$, illustrates the general situation quite well. More details on the present example are given in [109], and similar examples with more nodes are studied in [121].

Let the quadrature nodes and weights be $a_j = \omega^j$ and $c_j = \pi r^2$ respectively ($j = 1, 2, 3$), where $\omega = \mathrm{e}^{2\pi\mathrm{i}/3}$ and where $r > 0$ is a parameter. Considering first the strongest form of quadrature property, namely for subharmonic functions, as in (173), (174), the situation is in principle easy: Ω is for any given $r > 0$ uniquely determined up to nullsets and can be viewed as a swept out version of the quadrature measure $\mu = \sum_{j=1}^{3} c_j \delta_{a_j}$ or as the union of the discs $B(a_j, r)$ with (possible) multiple coverings smashed out.

For $0 < r \leqslant \frac{\sqrt{3}}{2}$ the above discs are disjoint, hence $\Omega = \cup_{j=1}^{3} B(a_j, r)$. For r larger than $\frac{\sqrt{3}}{2}$ but smaller than a certain critical value r_0 (which seems to be difficult to determine explicitly) Ω is doubly connected with a hole containing the origin, while for $r \geqslant r_0$ the hole will be filled in so that Ω is a simply connected domain. The above quadrature domains (or open sets) are actually uniquely determined even within nullsets, except in the case $r = r_0$ when both Ω and $\Omega\backslash\{0\}$ satisfy (173).

Consider next the general class of quadrature domains for analytic functions (algebraic domains). For $0 < r \leqslant \frac{\sqrt{3}}{2}$ only the disjoint discs qualify, as before. However, for any $r > \frac{\sqrt{3}}{2}$ there is a whole one-parameter family of domains Ω satisfying the quadrature identity for analytic φ. These are defined by the polynomials

$$\begin{aligned} Q(z, \bar{z}) = {} & z^3\bar{z}^3 - z^3 - \bar{z}^3 - 3r^2 z^2 \bar{z}^2 \\ & - 3\tau(\tau^3 - 2r^2\tau + 1) z\bar{z} + \tau^3(2\tau^3 - 3r^2\tau + 1), \end{aligned} \tag{188}$$

where $\tau > 0$ is a free parameter, independent of the quadrature data. When completed as to nullsets, the quadrature domains in question are more precisely

$$\Omega(r, \tau) = \operatorname{intclos}\{z \in \mathbb{C} : Q(z, \bar{z}) < 0\}.$$

The interpretation of the parameter τ is that on each radius $\{z = t\omega^{j+\frac{1}{2}} : t > 0\}$, $j = 1, 2, 3$, there is exactly one singular point of the algebraic curve $Q(z, \bar{z}) = 0$, and $\tau = |z|$ for that point. This singular point is either a cusp on $\partial\Omega$ or an isolated point of $Q(z, \bar{z}) = 0$, a so-called special point. Special points are those points $a \in \Omega$ for which the quadrature identity admits the (integrable) meromorphic function $\varphi(z) = \frac{1}{z-a}$. Equivalently, $\Omega\backslash\{a\}$ remains to be a quadrature domain for integrable analytic functions.

For $\frac{\sqrt{3}}{2} < r < 2^{-\frac{1}{6}}$ the quadrature domains for analytic functions are exactly the domains $\Omega(r, \tau)$ (with possible removal of special points) for τ in an interval $\tau_1(r) \leqslant \tau \leqslant \tau_2(r)$, where $\tau_1(r), \tau_2(r)$ satisfy $0 < \tau_1(r) < \frac{1}{2} < \tau_2(r)$, and more precisely can be defined as the positive zeros of the polynomial $4\tau^3 - 4r^2\tau + 1$. (see [109] for further explanations and proofs). The domains $\Omega(r, \tau)$ are doubly connected with a hole containing the origin. When τ increases the hole shrinks and both boundary components move toward the origin. For $\tau = \tau_2(r)$ there are three cusps on the outer boundary component which stop further shrinking of the hole, and for $\tau = \tau_1(r)$ there are three cusps on the inner boundary component which stop the expansion of the hole.

For exactly one parameter value, $\tau = \tau_{\text{subh}}(r)$, $\Omega(r,\tau)$ is a quadrature domain for subharmonic functions (and so also for harmonic functions). This $\tau_{\text{subh}}(r)$ can be determined implicitly by evaluating the quadrature identity for $\varphi(z) = \log|z|$, which gives the equation

$$\int_{\Omega(r,\tau_{\text{subh}}(r))} \log|z|\,\mathrm{dA}(z) = 0.$$

For $r = \frac{\sqrt{3}}{2}$, $\tau_1(r) = \tau_2(r) = \frac{1}{2}$, and as r increases, $\tau_1(r)$ decreases and $\tau_2(r)$ increases. What happens when $r = 2^{-\frac{1}{6}}$ is that for $\Omega(r,\tau_2(r))$, i.e., for the domain with cusps on the outer component, the hole has shrunk to a point (the origin). Hence, for $r = 2^{-\frac{1}{6}}$, $\Omega(r,\tau_2(r))$ is simply connected, while $\Omega(r,\tau)$ for $\tau_1(r) \leqslant \tau < \tau_2(r)$ remain doubly connected.

For all $\frac{\sqrt{3}}{2} < r \leqslant 2^{-\frac{1}{6}}$, $\tau_1(r) < \tau_{\text{subh}}(r) < \tau_2(r)$ because a subharmonic quadrature domain cannot have the type of cusps which appear for $\tau = \tau_1(r), \tau_2(r)$ (see [99, 100]). It follows that the critical value $r = r_0$, when $\Omega(r,\tau_{\text{subh}}(r))$ becomes simply connected, is larger that $2^{-\frac{1}{6}}$.

For $r \geqslant 2^{-\frac{1}{6}}$ the quadrature domains for analytic functions are the domains $\Omega(r,\tau)$ (with possible deletion of special points), with τ in an interval $\tau_1(r) \leqslant \tau \leqslant \tau_3(r)$. Here $\tau_1(r)$ is the same as before (i.e., corresponds to cusps on the inner boundary), while $\tau_3(r)$ is the value of τ for which the hole at the origin degenerates to just the origin itself (which for $r > 2^{-\frac{1}{6}}$ occurs before cusps have developed on the outer boundary). The origin then is a special point, and one concludes from (188) that $\tau = \tau_3(r)$ is the smallest positive zero of the polynomial $2\tau^3 - 3r^2\tau + 1$. For $r = 2^{-\frac{1}{6}}$, $\tau_3(r) = \tau_2(r) = 2^{-\frac{2}{3}}$.

For $2^{-\frac{1}{6}} \leqslant r < r_0$ we have $\tau_1(r) < \tau_{\text{subh}}(r) < \tau_3(r)$, while for $r \geqslant r_0$, $\tau_{\text{subh}}(r) = \tau_3(r)$. Since $\Omega(r,\tau_3(r))$ is simply connected and is a quadrature domain for analytic functions it is also a quadrature domain for harmonic functions. It follows that in the interval $2^{-\frac{1}{6}} \leqslant r < r_0$ there are (for each r) two different quadrature domains for harmonic functions, namely $\Omega(r,\tau_{\text{subh}}(r))$ and $\Omega(r,\tau_3(r))$ (doubly respectively simply connected).

In summary, we have for each $r > \frac{\sqrt{3}}{2}$ a one-parameter family of algebraic domains $\Omega(r,\tau)$, for exactly one parameter value ($\tau = \tau_{\text{subh}}(r)$) this is a quadrature domain for subharmonic functions, and for each r in a certain interval ($2^{-\frac{1}{6}} \leqslant r < r_0$) there are two different quadrature domains for harmonic functions ($\Omega(r,\tau_{\text{subh}}(r))$ and $\Omega(r,\tau_3(r))$).

6. Other physical applications of the operator theory formulation

The preceding sections provide a review of the relationships between the theory of normal random matrices, where evolution is defined by increasing the size of the matrix (a discrete time), its continuum (or infinite size) limit, Laplacian growth, and the general theory of semi-normal operators whose spectrum approximates generic domains. The exposition reflects, to some extent, the parallel historical development of the two non-commutative generalizations of Laplacian growth (random matrix theory and semi-normal operator theory). It is quite natural, at this point, to investigate the direct relationships between these two theories. However, this is a task of a magnitude which would require a separate review at the very least. We will therefore contend ourselves with exposing only a few of these relations, via their applications to physical problems.

The first application has to do with refined asymptotic expansions which characterize Laplacian growth in the critical case, before formation of a (2, 3) cusp. As we will see, to obtain this limit, one must take a 'double-scaling limit' by fine-tuning two parameters of the random matrix ensemble. Alternatively, this procedure is equivalent to a special choice of Padé approximants in the operator theory approach.

The second application described in this section is a very brief introduction of the notion of free, non-commutative random variables, and its relevance in open problems of strongly interacting quantum models, particularly in the 2D metal–insulator transition and the determination of ground state for 2D spin models. The review concludes with this cursory exposition.

6.1. Cusps in Laplacian growth: Painlevé equations

In this section, we exploit the formalism built up to now, in order to address a problem of great significance both at the mathematical and physical levels: what happens when a planar domain evolving under Laplacian growth approaches a generic (2, 3) cusp? We have already seen that a *classical* solution does not exist, in that no singly-connected domain with uniform density would satisfy the conditions of the problem. However, since we now have alternative formulations of Laplacian growth via the *balayage* of the uniform measure, we may generalize the problem and ask whether there is *any* equilibrium measure, dropping the uniformity (and indeed, the two-dimensional support) of the classical solution. By analogy with the 1D situation, we seek a solution in the sense of Saff and Totik, where the support and density of the equilibrium measure are given by the proper weighted limit of orthogonal polynomials. In order to obtain this limit, we must organize the evolution equations of the wavefunction such as to extract the correct scaling limit, for $N \to \infty$.

6.1.1. Universality in the scaling region at critical points—a conjecture. Detailed analysis of critical Hermitian ensembles indicates that the behavior of orthogonal polynomials in a specific region including the critical point (the *scaling region*), upon appropriate scaling of the degree n, is essentially independent of the bulk features of the ensemble. This *universality* property (a common working hypothesis in the physics of critical phenomena) is expected to occur for critical NRM ensembles as well—and is indeed easy to verify in critical Gaussian models, $2|t_2| = 1$. Analytically, it means that by suitable scaling of the variables z, n,

$$n \to \infty, \qquad \hbar \to 0, \qquad n\hbar = t_0, \qquad t_0 = t_c - \hbar^{\delta}\nu, \qquad z = z_c + \hbar^{\epsilon}\zeta,$$

where z_c is the location of the critical point and t_c is the critical area, the wavefunction $\Psi_n(z)$ will reveal a universal part $\phi(\nu, \zeta)$ which depends exclusively on the local singular geometry $x^p \sim y^q$ (p, q mutual primes) of the complex curve at the critical point. This conjecture is a subject of active research. Its main consequence is that in order to describe the scaling behavior for a certain choice of p, q, it is possible to replace a given ensemble with another which leads to the same type of critical point, though they may be very different at other length scales.

6.1.2. Scaling at critical points of normal matrix ensembles. In the remainder of the section we analyze the regularization of Laplacian growth for a critical point of type $p = 3, q = 2$, by discretization of the conformal map as described in the previous paragraph. For simplicity, we start from the conformal map corresponding to the potential $V(z) = t_3 z^3$, which is the simplest model leading to the specified type of cusp. It should be noted that the analysis will be identical for any monomial potential $V(z) = t_n z^n, n \geqslant 3$; for every such map, n singular points of type $p = 3, q = 2$ will form simultaneously on the boundary. The critical boundary corresponding to $n = 3$ is shown in figure 7.

The scaling limit from the string equation. We start from the Lax pair corresponding to the potential $V(z) = t_3 z^3$,

$$L\psi_n = r_n\psi_{n+1} + u_n\psi_{n-2}, \qquad L^\dagger\psi_n = r_{n-1}\psi_{n-1} + \bar{u}_{n+2}\psi_{n+2}. \tag{189}$$

The string equation (57) $[L^\dagger, L] = \hbar$ translates into

$$\begin{aligned}&\left(r_n^2 + |u|_n|^2 - r_{n-1}^2 - |u|_{n+2}^2\right)\psi_n + (r_n\bar{u}_{n+3} - r_{n+2}\bar{u}_{n+2})\psi_{n+3}\\ &\quad + (r_{n-3}u_n - r_{n-1}u_{n-1})\psi_{n-3} = \hbar\psi_n.\end{aligned} \tag{190}$$

Identifying the coefficients gives

$$\left(r_n^2 - |u|_{n+2}^2 - |u|_{n+1}^2\right) - \left(r_{n-1}^2 - |u|_{n+1}^2 - |u|_n^2\right) = \hbar, \tag{191}$$

and

$$\frac{\bar{u}_{n+2}}{r_n r_{n+1}} = \frac{\bar{u}_{n+3}}{r_{n+1}r_{n+2}} = 3t_3. \tag{192}$$

Equation (191) gives the quantum area formula

$$r_n^2 - \left(|u|_{n+2}^2 + |u|_{n+1}^2\right) = n\hbar, \tag{193}$$

which together with the conservation law (192) leads to the discrete Painlevé equation

$$r_n^2\left[1 - 9|t_3|^2\left(r_{n-1}^2 + r_{n+1}^2\right)\right] = n\hbar. \tag{194}$$

In the continuum limit, the equation becomes

$$r^2 - 18|t_3|^2 r^4 = t_0. \tag{195}$$

The critical (maximal) area is given by

$$\frac{\mathrm{d}t_0}{\mathrm{d}r^2} = 0, \qquad 36|t_3|^2 r_c^2 = 1. \tag{196}$$

Choosing $r_c = 1$ gives $6|t_3| = 1$ and $t_c = \frac{1}{2}$. It also follows that

$$u_n = \frac{r_{n-2}r_{n-1}}{2}, \qquad z_c = \frac{3}{2}. \tag{197}$$

Introduce the notations

$$N\hbar = t_c, \qquad n\hbar = t_0 = t_c + \hbar^{4a}\nu, \qquad r_n^2 = 1 - \hbar^{2a}u(\nu), \qquad z = \tfrac{3}{2} + \hbar^{2a}\zeta, \tag{198}$$

where $a = \frac{1}{5}$. We get $\partial_n = \hbar^a\partial_\nu$ and

$$r_{n+k}^2 = 1 - \hbar^{2a}u - k\hbar^{3a}\dot{u}(\nu) - \frac{k^2}{2}\hbar^{4a}\ddot{u}, \tag{199}$$

where dot signifies derivative with respect to ν. The scaling limit of the quantum area formula becomes

$$(1 - \hbar^{2a}u)\left[\frac{1}{2} + \hbar^{2a}\frac{u}{2} + \hbar^{4a}\frac{\ddot{u}}{4}\right] = \frac{1}{2} + \hbar^{4a}\nu, \tag{200}$$

giving at order $\hbar^{4a}$ the Painlevé I equation

$$\ddot{u} - 2u^2 = 4\nu. \tag{201}$$

Rescaling $u \to c_2 u$, $\nu \to c_1\nu$ gives the standard form

$$\ddot{u} - 3u^2 = \nu, \tag{202}$$

for $c_2 = 4c_1^3, 8c_1^5 = 3$.

Painlevé I as compatibility equation. Inspired by the Saff–Totik approach, we construct the wavefunctions based on monic polynomials (Pol is the polynomial part),

$$\phi_n = \prod_{i=0}^{n-1} r_i \psi_n, \qquad \mathrm{Pol}\, \phi_n(z) = z^n + O(z^{n-1}), \tag{203}$$

and rewrite the equations for the Lax pair as

$$L\phi_n = \phi_{n+1} + \frac{r_{n-2}^2 r_{n-1}^2}{2}\phi_{n-2}, \qquad L^\dagger \phi_n = r_{n-1}^2 \phi_{n-1} + \frac{\phi_{n+2}}{2}. \tag{204}$$

Note that using the shift operator $\mathcal{W}$, the system can also be written as

$$L = \mathcal{W} + \tfrac{1}{2}\left(r_{n-1}^2 \mathcal{W}^{-1}\right)^2, \qquad L^\dagger = r_{n-1}^2 \mathcal{W}^{-1} + \tfrac{1}{2}\mathcal{W}^2. \tag{205}$$

Introduce the scaling ψ function through

$$\phi_n(z) = \mathrm{e}^{\frac{z^2}{2\hbar}} \psi(\zeta, \nu). \tag{206}$$

The action of Lax operators on ψ gives the representation

$$L = \tfrac{3}{2} + \hbar^{2a}\zeta, \qquad L^\dagger = z + \hbar\partial_\zeta = \tfrac{3}{2} + \hbar^{2a}\zeta + \hbar^{3a}\partial_\zeta. \tag{207}$$

Therefore, the action of ζ is given by the sum of equations at order $\hbar^{2a}$,

$$3 + 2\hbar^{2a}\zeta = \mathcal{W} + \tfrac{1}{2}\mathcal{W}^2 + r_{n-1}^2\mathcal{W}^{-1} + \tfrac{1}{2}\left(r_{n-1}^2 \mathcal{W}^{-1}\right)^2, \tag{208}$$

and the action of ∂_ζ by their difference

$$\hbar^{3a}\partial_\zeta = -\mathcal{W} + \tfrac{1}{2}\mathcal{W}^2 + r_{n-1}^2\mathcal{W}^{-1} - \tfrac{1}{2}\left(r_{n-1}^2 \mathcal{W}^{-1}\right)^2. \tag{209}$$

Equivalently, we can write

$$\hbar^{2a}\zeta = \tfrac{1}{2}\left[\left(\mathcal{W}+1\right)^2 + \left(r_{n-1}^2\mathcal{W}^{-1}+1\right)^2\right] - 4, \tag{210}$$

$$\hbar^{3a}\partial_\zeta = \tfrac{1}{2}\left[\left(\mathcal{W}-1\right)^2 - \left(r_{n-1}^2\mathcal{W}^{-1}-1\right)^2\right]. \tag{211}$$

Expanding the shift operator in $\hbar$ leads to

$$\mathcal{W} = 1 + \hbar^a\partial_\nu + \hbar^{2a}\frac{\partial_\nu^2}{2} + \hbar^{3a}\frac{\partial_\nu^3}{6}, \tag{212}$$

and

$$r_{n-1}^2\mathcal{W}^{-1} = 1 - \hbar^a\partial_\nu + \hbar^{2a}\left(\frac{\partial_\nu^2}{2} - u\right) + \hbar^{3a}\left(-\frac{\partial_\nu^3}{6} + u\partial_\nu + \dot{u}\right). \tag{213}$$

Substituting into the equations for ζ, ∂_ζ gives the system of equations

$$\ddot{\psi} = \frac{2(\zeta+u)}{3}\psi, \qquad \psi' = \frac{\dot{u}}{6}\psi + \frac{2\zeta - u}{3}\dot{\psi}, \tag{214}$$

where primed variables are differentiated with respect to ζ. The equations can be written in matrix form as

$$\Psi' = \Lambda\Psi, \qquad \dot{\Psi} = Q\Psi, \qquad \Psi = \begin{pmatrix}\psi \\ \dot{\psi}\end{pmatrix}, \tag{215}$$

where

$$\Lambda = \begin{pmatrix} \frac{\dot{u}}{6} & \frac{2\zeta - u}{3} \\ \frac{\ddot{u}}{6} + \frac{2(\zeta + u)(2\zeta - u)}{9} & -\frac{\dot{u}}{6} \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 1 \\ \frac{2(\zeta + u)}{3} & 0 \end{pmatrix}. \tag{216}$$

The compatibility equations

$$\dot{\Lambda} - Q' = [Q, \Lambda] \tag{217}$$

yield the Painlevé equation derived in the previous section

$$\dot{\Lambda} = \begin{pmatrix} \frac{\ddot{u}}{6} & -\frac{\dot{u}}{3} \\ \frac{\dddot{u}}{6} + \frac{2\zeta\dot{u} - 4u\dot{u}}{9} & -\frac{\ddot{u}}{6} \end{pmatrix}, \qquad Q' = \begin{pmatrix} 0 & 0 \\ \frac{2}{3} & 0 \end{pmatrix}, \tag{218}$$

and

$$[Q, \Lambda] = \begin{pmatrix} \frac{\ddot{u}}{6} & -\frac{\dot{u}}{3} \\ \frac{2(\zeta + u)\dot{u}}{9} & -\frac{\ddot{u}}{6} \end{pmatrix}. \tag{219}$$

Thus,

$$0 = \dot{\Lambda} - Q' - [Q, \Lambda] = \begin{pmatrix} 0 & 0 \\ \frac{\dddot{u}}{6} - \frac{6u\dot{u}}{9} - \frac{2}{3} & 0 \end{pmatrix}. \tag{220}$$

The only non-trivial element of the matrix gives

$$\dddot{u} - 4u\dot{u} - 4 = 0, \tag{221}$$

i.e. the Painlevé equation derived in the previous section.

6.1.3. Conclusions. The derivations presented above indicate that, in the vicinity of a (2, 3) cusp, the refined asymptotics for Laplacian growth are based on the behavior of the Baker–Akhiezer function for the Painlevé I equation. This fact allows us to properly define the evolution of the domain beyond the critical time, by identifying the support of the measure with the support of the zeros of this function. This is a work in progress which will be reported elsewhere.

It is also interesting to note that the double-scaling limit required to derive the refined asymptote mirrors an earlier result, due in its original form to Stahl [150], and related to orthogonal polynomials in [151]. It describes an approximation of the Cauchy transform of a planar domain via a special sequence of Padé approximants (in the spirit of section 5.3.2), which by exponentiation would translate into the double-scaling limit presented in this section.

6.2. Non-commutative probability theory and 2D quantum models

We conclude this review with a brief presentation of outstanding problems in two-dimensional quantum models, where the use of random matrix theory led to important results, and (perhaps most importantly) pointed out to the need for a probability theory for *non-commutative* random variables. In turn, such a theory is intimately related to the semi-normal operator approach presented in the previous section.

6.2.1. Metal–insulator transition in two dimensions. The details of the transition from conductive to insulating behavior for a system of interacting 2D electrons, in the presence of disorder, referred to as metal–insulator transition (or MIT) are not well understood, despite decades of research. Here we give a very sketchy description of this problem, in order to illustrate the mathematical essence of the model and of the difficulties, and we refer the reader to one of the several excellent monographs on the subject [152]. The fact that a system of electrons may 'jam', i.e. behave like an insulator, because of either strong interactions (Mott transition) or strong disorder (Anderson transition) has been known for roughly half a century. However, creating a theoretical model which could incorporate both interactions and disorder in a proper fashion, was difficult to achieve. The foundation for our current formulation of this problem was laid by Wegner [14], and later improved by Efetov [15]. A very clear exposition of this formulation can be found in the synopsis [16].

In its simplest formulation, the model consists of a lattice in d-dimensions (which may be taken to be $\mathbb{Z}^d$), where to each vertex corresponds an n-dimensional vector space of states (also called orbitals), and with Hamiltonian

$$H = H_0 + H_d, \qquad H_0 = \sum_{n,\langle x,y\rangle} t_{x,y} |x, n\rangle\langle y, n|, \qquad H_d = \sum_{x,i,j} f^{ij} |x, i\rangle\langle x, j|,$$

where the state $|n, x\rangle$ depends on position x and orbital n, H_0 refers to the interaction between adjacent vertices and H_d implements the disorder component, via the random matrix f^{ij}, which can be Hermitian, Orthogonal, Symplectic, etc based on symmetries of the system. Efetov's idea was to use supersymmetry to incorporate interactions and disorder on the same footing; the method was later extended to implement the 'Hermitization' of non-Hermitian random matrices with non-Gaussian weights, appearing in the same physical context [153]. For rotationally invariant measures, the authors showed that the distribution of eigenvalues can be either a disc or an annulus, and that there is a phase transition between the two, as a function of model parameters.

The difficulties related to this formulation of the problem are due to the fact that the transition cannot be described within the established models of phase transitions. In all these models, the state of the system is obtained by minimization of a proper thermodynamic potential (for instance, free energy), or equivalently, finding the points of extrema of action in a path-integral approach (via a saddle-point condition). 'Proper' phase transitions are characterized by potentials that are globally convex, so that the minimization problem is well defined. However, the supersymmetric formulation of MIT does not lead to a true extremum, but rather a saddle-point, due to the non-compact, hyperbolic geometry structure of the effective theory ($SU(1, 1)$ in the simplest case). The interested reader can find a detailed exposition of this phenomenon in [154], for example. In the case where the system has a finite scale, it can be shown, following Efetov, that only the zero modes of the theory are important, which leads to an effective simplification in computing the multipoint correlation functions. However, the full model is still not solved for the 2D case, due to the difficulties pointed out above.

In a nutshell, we may summarize the problem as non-tractable using the standard statistical physics formulation of phase transitions. In that sense, the situation is similar to another famous unsolved physical model, the disordered spin problem in the presence of magnetic field, where determination of the ground state is a task of exponential complexity (with respect to the size of the system). The phase transition where the system goes from an ordered state to a state with local order but no long-range order (a spin glass) is equally intractable as MIT, for the reasons explained.

Interestingly enough, both problems may be *approximately* studied using a physicist's approach notorious for its lack of control: the replica-symmetry breaking (RSB) [155]. We mention it here mainly because of its statistical interpretation.

Starting from the elementary observation $\log Z = \lim_{n\to 0}(Z^n - 1)/n$, it is tempting to replace averages (over disorder) computed from the thermodynamic potential $\langle \log Z\rangle$, with averages computed with $\langle Z^n\rangle$, because of the implicit assumption that repeated products of the random variable Z will *self-average* (an implicit application of the central limit theorem). By extension, one may assume that averages of products of operators, projected on special states, $\langle 0|\phi_1\phi_2 \ldots \phi_k|0\rangle$ (correlation functions), may also be computed by the same argument.

In this (statistical inference) approach, the failure of standard descriptions of phase transitions is related to reducing correlation functions of products of operators, to their *projections* onto selected states. At the critical point, such projections do not have the expected convergence properties. It is therefore natural to ask whether one may use other (weaker) criteria to determine the critical point. In particular, is it possible to define statistical inference for the operators themselves, rather than special projections?

The answer is affirmative, and such a theory was constructed almost in parallel with the MIT and spin-glass models described above.

6.2.2. Non-commutative probability theory and free random variables. The basic elements in the probability theory for non-commutative operators [156] are the following: $\mathcal{A}$, a non-commutative (operator) algebra over $\mathbb{C}$, $1 \in \mathcal{A}$; a functional $\phi : \mathcal{A} \to \mathbb{C}$, $\phi(1) = 1$, called expectation functional.

Quantum mechanics offers specific examples:

- Example 1: $\mathcal{A}$ = bounded operators over Hilbert space of states $\mathcal{H}$, $\xi \in \mathcal{H}$, $\|\xi\| = 1$, the ground state, and

$$\phi(A) = \langle \xi|A|\xi\rangle.$$

- Example 2: $\mathcal{A}$ = von Neumann algebra over $\mathcal{H}$, and functional $\phi = \mathrm{Tr}$.

In order to develop inference methods within this theory, it is necessary to define the equivalent of independent variables in commutative probability. Such variables are called *free*, and satisfy the following property: $A_1, A_2, \ldots, A_k$ are free if $\phi(A_i) = 0$, and

$$\phi(A_{i_1}A_{i_2}\ldots A_{i_k}) = 0, \qquad A_{i_j} \neq A_{i_{j+1}}.$$

Using these tools, generalizations of standard results in large sample theory are possible. We mention a few:

- The 'Gaussian' distribution (limit distribution for central limit theorem) in free probability theory is given by operators with eigenvalues obeying the semi-circle distribution (Wigner–Dyson) $\rho(\lambda) = \sqrt{a^2 - \lambda^2}$.
- Similarly, the Poisson distribution has as free correspondent the operators with eigenvalues distribution according to the Marchenko–Pastur (elliptical law), $\rho(\lambda) = \sqrt{(\lambda - a)(b - \lambda)}$.
- The free Cauchy distribution is the Cauchy distribution itself.

Likewise, there is a notion of free Fisher entropy, Cramér–Rao bound, etc.

As announced earlier, the relation between this theory, random matrices and operator theory for 2D spectral support is two-fold: on the one hand, we have the important result that random matrices, in the large-size limit, become free non-commutative random variables. Thus, inference in free non-commutative probability may be approximated using ensembles of random matrices, which explains the success of this concept in the physics of disordered quantum systems.

On the other hand, the limit distributions specified above (via Wigner–Dyson, Marchenko–Pastur laws and their 2D counterparts) are described through spectral data. Taking this as a starting point, it is relevant to construct sequences of operators which approximate the spectrum, which points directly to the methods of section 5.

As a last remark, an early attempt to employ non-commutative probability theory in MIT was reported in [157]. It is likely that the application of this generalized inference method will help elucidate open questions like the ones discussed in this section.

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